CS / MCS 401 Homework 1 grader solutions

assignment due June 20, 2016
written by Jānis Lazovskis
maximum points: 16

Questions from Appendix A.1, page 1149 in CLRS

A.1-1 *Find a simple formula for* $\sum_{k=1}^{n}(2k-1)$.

We find the answer by splitting up the summand as

$$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2 \cdot \frac{n(n+1)}{2} - n = n^2.$$

■

A.1-2 *Show that* $\sum_{k=1}^{n} 1/(2k-1) = \ln(\sqrt{n}) + O(1)$ *by manipulating the harmonic series.*

Rewrite the given series as the difference of two other series, by expanding out the sum:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{2k-1} &= \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1} \\
&= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{2n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{2k} \\
&= \ln(2n) + O(1) - \frac{1}{2}\ln(n) + O(1) \\
&= \ln(2) + \ln(n) - \frac{1}{2}\ln(n) + O(1) \\
&= \frac{1}{2}\ln(n) + O(1) \\
&= \ln(\sqrt{n}) + O(1).
\end{aligned}$$

Note that two expressions "$O(1)$" combine into one, and the constant $\ln(2)$ is also absorbed into it. ■

A.1-3 *Show that* $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ *for* $0 < |x| < 1$.

From the textbook we are given that $\sum_{k=0}^{\infty} kx^k = x/(1-x)^2$, so differentiating both sides with respect to x gives

$$\begin{aligned}
\frac{d}{dx}\left(\sum_{k=0}^{\infty} kx^k\right) &= \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) \\
\sum_{k=0}^{\infty} \frac{d}{dx}(kx^k) &= \frac{(1-x)^2 - x \cdot 2(1-x) \cdot (-1)}{(1-x)^4} \\
\sum_{k=1}^{\infty} k^2 x^{k-1} &= \frac{1-x+2x}{(1-x)^3} \\
\sum_{k=0}^{\infty} k^2 x^{k-1} &= \frac{1+x}{(1-x)^3}.
\end{aligned}$$

Note that when differentiating, we lose the $k=0$ term, but since it is zero in this case, we may add it back in without changing the value of the expression. Now multiply both sides of the last line above by x to get

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{x(1+x)}{(1-x)^3}.$$

■

$\boxed{\textbf{A.1-4}}$ *Show that* $\sum_{k=0}^{\infty}(k-1)/2^k = 0$.

Writing down the first few terms we may express the given sum differently, as

$$\sum_{k=0}^{\infty}\frac{k-1}{2^k} = -1 + 0 + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots = -1 + \frac{1}{2}\sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k.$$

Since $0 < 1/2 < 1$, we use the known formula for $\sum_{k=0}^{\infty} kx^k$ to get that

$$\sum_{k=0}^{\infty}\frac{k-1}{2^k} = -1 + \frac{1}{2}\cdot\frac{1/2}{(1-1/2)^2} = -1 + \frac{1/4}{1/4} = 0,$$

as desired. ■

$\boxed{\textbf{A.1-5}}$ *Evaluate the sum* $\sum_{k=1}^{\infty}(2k+1)x^{2k}$.

Notice that the coefficient of x^{2k} is one greater than its exponent, so we make an educated guess that the answer will involve a derivative or an integral. Indeed, first note that

$$\sum_{k=0}^{\infty} x^{2k+1} = x\sum_{k=0}^{\infty}(x^2)^k = \frac{x}{1-x^2}$$

for all $|x^2| < 1$ (or equivalently, for $|x| < 1$). Take the derivative of the left and the right side (noting that we stay at $k = 0$ because the first term $x^{2\cdot 0+1} = x$ is not a constant) to get

$$\sum_{k=0}^{\infty}(2k+1)x^{2k} = \frac{1-x^2-x\cdot(-2x)}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}.$$

The zeroth term is $(2\cdot 0+1)(x^{2\cdot 0}) = 1$, so

$$\sum_{k=1}^{\infty}(2k+1)x^{2k} = \frac{1+x^2}{(1-x^2)^2} - 1 = \frac{1+x^2-(1-x^2)^2}{(1-x^2)^2} = \frac{3x^2-x^4}{(1-x^2)^2}.$$

■

$\boxed{\textbf{A.1-6}}$ *Prove that* $\sum_{k=1}^{n} O(f_k(i)) = O(\sum_{k=1}^{n} f_k(i))$ *by using the linearity property of summations.*

Consider first the case with two functions f_1 and f_2. Recall the definition of O notation (on page 47 of CLRS), which says that for some constants c_1, c_2, n_1, n_2,

$$\begin{aligned} O(f_1) &= \{g \ : \ 0 \leqslant g(n) \leqslant c_1 \cdot f_1(n) \text{ for all } n \geqslant n_1\}, \\ O(f_2) &= \{h \ : \ 0 \leqslant h(n) \leqslant c_2 \cdot f_2(n) \text{ for all } n \geqslant n_2\}. \end{aligned}$$

Let $N = \max\{n_1, n_2\}$ and $C = \max\{c_1, c_2\}$, so the definitions of $O(f_1)$ and $O(f_2)$ hold for the constants C and N. Indeed, we have that

$$\begin{aligned} O(f_1) + O(f_2) &= \{g + h \ : \ 0 \leqslant g(n) \leqslant C\cdot f_1(n) \text{ and } 0 \leqslant h(n) \leqslant C \cdot f_2(n) \text{ for all } n \geqslant N\} \\ &\subseteq \{g + h \ : \ 0 \leqslant g(n) + h(n) \leqslant C(f_1(n) + f_2(n)) \text{ for all } n \geqslant N\} \\ &= O(f_1 + f_2) \end{aligned}$$

by adding up the two inequalities. Hence $O(f_1) + O(f_2) \subseteq O(f_1 + f_2)$ *as sets*, but in big-O notation, we now say that $O(f_1) + O(f_2)$ *is* $O(f_1 + f_2)$, enough for us to say that $O(f_1) + O(f_2) = O(f_1 + f_2)$. This is similar to saying that $f = O(g)$ even though f is one of the many elements of the set $O(g)$.

Since the statement is true for 2 functions, by repeating the process n times (there is an unfortunate choice of index and function argument here), the result holds for a sum of n functions. ■

$\boxed{\textbf{A.1-7}}$ *Evaluate the product* $\prod_{k=1}^{n} 2 \cdot 4^k$.

First note that $2 \cdot 4^k = 2 \cdot 2^{2k} = 2^{2k+1}$. Then take the logarithm of this product to get

$$\begin{aligned}
\log\left(\prod_{k=1}^{n} 2 \cdot 4^k\right) &= \sum_{k=1}^{n} \log(2^{2k+1}) \\
&= \sum_{k=1}^{n} (2k+1)\log(2) \\
&= \log(2)\left(2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right) \\
&= \log(2)\left(2 \cdot \frac{n(n+1)}{2} + n\right) \\
&= \log(2)(n^2 + 2n) \\
&= \log(2^{n^2+2n}).
\end{aligned}$$

Taking the exponential of both sides (or comparing the arguments of logarithms), we get that

$$\prod_{k=1}^{n} 2 \cdot 4^k = 2^{n^2+2n}.$$

■

$\boxed{\textbf{A.1-8}}$ *Evaluate the product* $\prod_{k=2}^{n}(1-1/k^2)$.

By taking logarithms and factoring, we get

$$\begin{aligned}
\log\left(\prod_{k=2}^{n} 1-\frac{1}{k^2}\right) &= \sum_{k=2}^{n}\log\left(1-\frac{1}{k^2}\right)\\
&= \sum_{k=2}^{n}\log\left(\frac{k^2-1}{k^2}\right)\\
&= \sum_{k=2}^{n}\log\left(\frac{(k+1)(k-1)}{k^2}\right)\\
&= \sum_{k=2}^{n}\left(\log(k+1)+\log(k-1)-2\log(k)\right).
\end{aligned}$$

Consider the first few and last few terms, noting that there is cancellation that simplifies the sum:

$$\begin{array}{rccc}
k=2: & \log(3) & \log(1) & -2\log(2)\\
k=3: & \log(4) & \log(2) & -2\log(3)\\
k=4: & \log(5) & \log(3) & -2\log(4)\\
k=5: & \log(6) & \log(4) & -2\log(5)\\
 & \vdots & \vdots & \vdots\\
k=n-2: & \log(n-1) & \log(n-3) & -2\log(n-2)\\
k=n-1: & \log(n) & \log(n-2) & -2\log(n-1)\\
k=n: & \log(n+1) & \log(n-1) & -2\log(n)
\end{array}$$

The only leftover terms are

$$\log(1)-\log(2)+\log(n+1)-\log(n) = \log\left(\frac{1\cdot(n+1)}{2\cdot n}\right),$$

and using the same justification as above,

$$\prod_{k=2}^{n} 1-\frac{1}{k^2} = \frac{n+1}{2n}.$$

$\blacksquare$