BITL3

Worksheet 8

21 February 2022

Math Lab

1. **Warm up:** Answer the following True / False questions.
   (a) Every improper integral diverges.
   (b) Every definite integral converges.
   (c) The expression $\int_{-3}^{0} \frac{-1}{x^3}\, dx$ is an improper integral.
   (d) The expression in part (c) is an improper integral after $u = \frac{1}{x}$ substitution.
   (e) If f has a horizontal asymptote at $y = y_0$, then $\int_0^\infty f(x) - y_0\, dx$ is always finite.

2. Use the comparison test for integrals to determine if each of the following integrals converges or diverges.

$$\int_1^\infty \frac{\cos^4(x)}{x^4}\, dx \qquad\qquad \int_2^\infty 1x - 2^{-x}\, dx$$

3. Use integration by parts to show that $\int_1^\infty \frac{\sin(t)}{\sqrt{t}}\, dt$ is finite.

4. Let $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ for $x > 0$. This is called the *gamma function.*
   (a) Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.
   (b) Show that $\Gamma(1) = 1$.
   (c) Show that $\Gamma(n) = (n-1)!$ for all $n \in \mathbf{N}$ (the set of natural numbers).

5. Let f be a continuous function.
   (a) Show that if $\int_1^\infty f(x)\, dx$ diverges, then $\int_2^\infty f(x)\, dx$ diverges.
   (b) Is the converse to part (a) true? If yes, why? If no, find a counterexample.
   (c) Find an f for which $\int_2^\infty f(x)\, dx$ converges and $\int_1^\infty f(x)\, dx$ diverges.