BITL3

Worksheet 13

20 October 2021

Math Lab

If k is a constant and the limits $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$ exist, then

1. $\lim_{x\to a}[f(x)+g(x)] = A+B$

2. $\lim_{x\to a}[f(x)-g(x)] = A-B$

3. $\lim_{x\to a}[kf(x)] = kA$

4. $\lim_{x\to a}[f(x)\cdot g(x)] = A\cdot B$

5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{A}{B}$ if $B \neq 0$

6. $\lim_{x\to a}[f(x)]^n = A^n$

7. $\lim_{x\to a} k = k$

8. $\lim_{x\to a} x = a$

9. $\lim_{x\to a} x^n = a^n$

Asymptotes of functions are defined using limits:

- the function f has a **horizontal asymptote** at $y = a$ if $\lim_{x\to\infty} f(x) = a$ or $\lim_{x\to-\infty} f(x) = a$
- the function f has a **vertical asymptote** at $x = a$ if $\lim_{x\to a^\pm} f(x) = \pm\infty$.

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1. **Warm up:** Answer the following True / False questions about limits.

   (a) If $\lim_{x\to-2^+} f(x) \neq \lim_{x\to-2^-} f(x)$, then $f(-2)$ is not defined.

   (b) If $\lim_{x\to3} h(x)$ does not exist, then h does not have a tangent line at $x = 3$.

   (c) If h does not have a tangent line at $x = 3$, then $\lim_{x\to3} h(x)$ does not exist.

2. Evaluate the following limits.

   (a) $\lim_{x\to9} \dfrac{2x^2-3}{9x}$

   (b) $\lim_{y\to4} \dfrac{y^2-y-12}{\sqrt{y}-2}$

   (c) $\lim_{z\to3} \dfrac{3-z}{z-3}$

   (d) $\lim_{x\to0} x\sin(x)$

3. For each of the functions f below, identify their vertical asymptotes $x = a$, and evaluate the two limits $\lim_{x\to a^\pm} f(x)$.

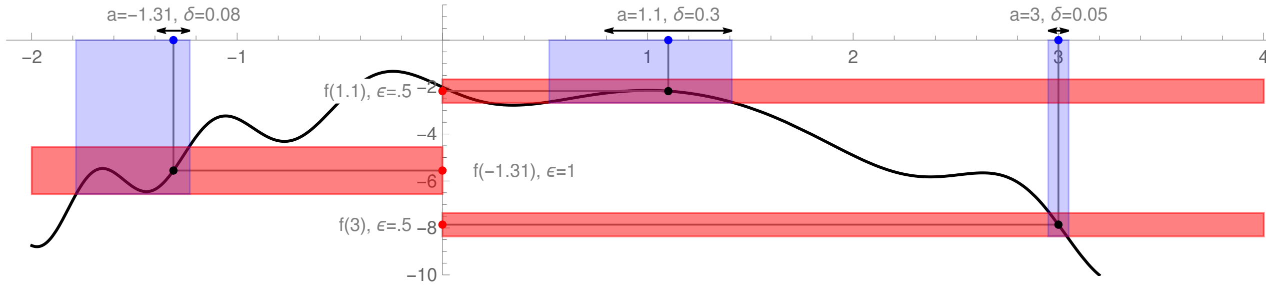
   (a) $\dfrac{x^2+3x-1}{x+2}$

   (b) $\dfrac{x^2-10x+16}{x-2}$

   (c) $\dfrac{3x}{(5+6x+x^2)^2}$

   (d) $\tan\left(\dfrac{\pi x}{2}\right)$

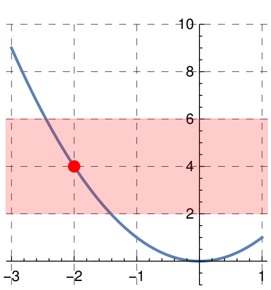
4. The idea of a *limit* is the following: Given a a function f and a number a, the limit of f at a exists is another way to say that for any arbitrarily small ϵ-neighborhood of $f(a)$ on the y-axis, there exists a sufficiently small δ-neighbourhood of a on the x-axis, so that when put together, these neigbbourhoods contain all pairs $(x, f(x))$.



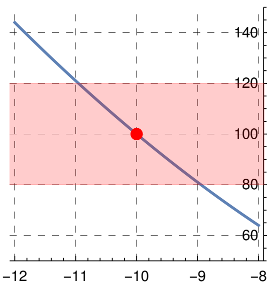
The arbitrarily means we don't control ϵ, and the sufficiently means we can control δ. In the example above, the δ-neighbourhood of $a = 1.1$ for $\delta = 0.3$ is $(a-\delta, a+\delta) = (0.8, 1.4)$. The value δ is usually given as large as possible.

You may use a calculator for the questions below.

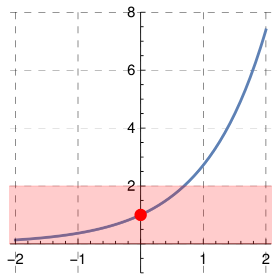
(a) For each of the following functions and numbers a, ϵ, find $\delta > 0$ so that $f(x)$ is in the ϵ-neigbourhood of $f(a)$ whenever x is in the δ-neighbourhood of a.



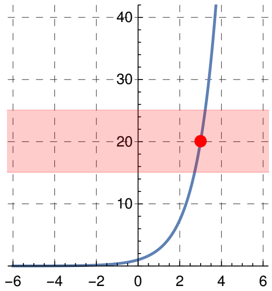
$$\begin{aligned} f(x) &= x^2 \\ a &= -2 \\ \epsilon &= 2 \end{aligned}$$



$$\begin{aligned} f(x) &= x^2 \\ a &= -10 \\ \epsilon &= 20 \end{aligned}$$

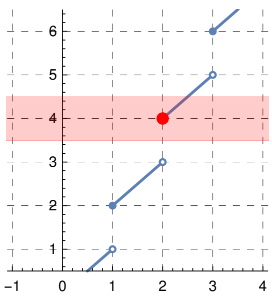


$$\begin{aligned} g(x) &= e^x \\ a &= 0 \\ \epsilon &= 1 \end{aligned}$$

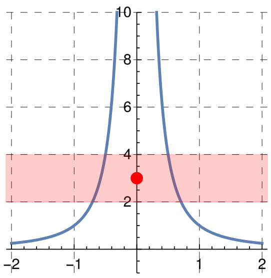


$$\begin{aligned} g(x) &= e^x \\ a &= 3 \\ \epsilon &= 5 \end{aligned}$$

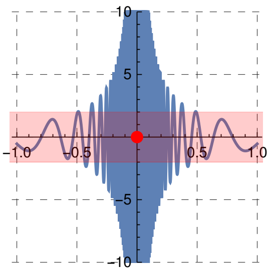
(b) For each of the following functions f and numbers a, ϵ, find a small $\delta > 0$ so that $f(x+\delta)$ or $f(x-\delta)$ is farther than ϵ from $f(a)$. This means the limit *does not exist* at the given number a.



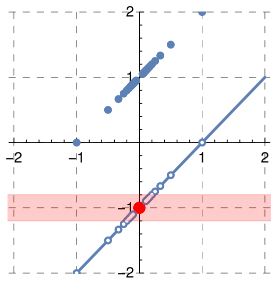
$$\begin{aligned} h(x) &= \lfloor x \rfloor + x \\ a &= 2 \\ \epsilon &= 0.5 \end{aligned}$$



$$\begin{aligned} k(x) &= \begin{cases} 1/x^2 & x \neq 0 \\ 3 & x = 0 \end{cases} \\ a &= 0 \\ \epsilon &= 1 \end{aligned}$$



$$\begin{aligned} \ell(x) &= \begin{cases} \frac{\sin(10/x)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \\ a &= 0 \\ \epsilon &= 2 \end{aligned}$$



$$\begin{aligned} m(x) &= \begin{cases} x-1 & x \neq \frac{1}{n} \\ x+1 & x = \frac{1}{n} \end{cases} \\ n &\in \mathbf{Z}_{\neq 0} \\ a &= 0 \\ \epsilon &= 0.2 \end{aligned}$$

(c) **Bonus:** For each of the functions in part (b), find the δ for an arbitrary $\epsilon > 0$.