BIT L3

Worksheet 6

23 September 2021

Math Lab

A **binomial** is a polynomial with two terms, such as $3x+4$ or $10x^2-5x^5$. The **binomial theorem** says what the expansion of the nth power of a binomial looks like, such as $(3x+4)^5$ for $n=5$ or $(1-x^2-5x^5)^{15}$ for $n=15$:

$$(a+b)^n = \sum_{i=1}^{n} \binom{n}{k} a^{n-k}b^k = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

This is useful when you are asked what the **coefficient** of x^k will be in the expansion of $(ax+b)^n$, for some $k=1,2,\ldots,n$. By the binomial theorem, the coefficient will be

$$a^{n-k}\cdot b^k \cdot \binom{n}{k} = \frac{a^{n-k}\cdot b^k\cdot n!}{k!(n-k)!}.$$

A **sum** is a number that results from adding other numbers, called **summands**. The **summation symbol** Σ indicates a very long sum whose summands follow a particular pattern:

$$\sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = 1+4+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

The method of **mathematical induction** is way to prove general statements that depend on some number $n\in\mathbf{N}$. If the statement is $P(n)$, the method is as follows:

1. (*Base case*) Show that $P(1)$ is true.
2. (*Inductive hypothesis*) Assume that $P(k)$ is true for $k \geqslant 1$
3. (*Inductive step*) Show that $P(k+1)$ is true

Sometimes the base case uses a number larger than 1, because $P(n)$ only holds for $n\geqslant 4$.

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1. **Warm up:** What is the coefficient of x^5 in each of the following cases? Do not simplify.

$$(2x+1)^{10} \qquad \left(\frac{x}{5}-5\right)^{55} \qquad (1+2x+3+4x)^7$$

2. Using the definition of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, prove the following statements.

$$\binom{n}{k} = \binom{n}{n-k} \qquad \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

3. Evaluate the following sums.

$$\sum_{j=1}^{2n} 3j^2+4 \qquad \sum_{i=100}^{200} 4-5i \qquad \sum_{i=1}^{n}\sum_{j=1}^{m} 3ij+2i-5j^2$$

4. Let $P(n)$ be the following statement: $n! > 2^n$
   (a) Check if the statements $P(1)$, $P(2)$, $P(3)$, $P(4)$, $P(5)$ are true or false.
   (b) Prove the statement by induction, for $n\geqslant 4$.

5. **Bonus:** Using induction, prove that $2^{2n-1}+3^{2n-1}$ is always divisible by 5, for any $n\in\mathbf{N}$.