BITL3 **Math Lab**

Worksheet 5

22 September 2021

Recall the big theorems related to a polynomial $p(x)$:

- **Remainder theorem:** Dividing $p(x)$ by $x - k$ leaves the remainder $p(k)$
- **Factor theorem:** $k \in \mathbf{R}$ is a root of $p(x)$ iff dividing $p(x)$ by $x - k$ leaves the remainder 0
- **Integer root theorem:** $k \in \mathbf{Z}$ is a root of $p(x)$ only if k divides the constant term of $p(x)$
- **Rational root theorem:** $\frac{a}{b} \in \mathbf{Q}$ is a root of $p(x)$ only if a divides the constant term of $p(x)$ and b divides the leading coefficient of $p(x)$

You will not be tested on the last theorem in homeworks / exams.

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1. **Warm up:** Use the remainder theorem to determine the value of $p(k)$ in each case.

$$\frac{p(x)}{x-k} = 3x^2 + 2x - 4 + \frac{7}{x-k} \qquad \frac{p(x)}{x} = 10x - 10x^{-1} \qquad \frac{p(x)}{x-k} = 99 - 99x^2$$

2. For each of the following polynomials $p(x)$, use the integer root theorem to find all potential integer roots k, then check if $p(k) = 0$ and use the factor theorem to determine the roots.

   (a) $x^2 - 8x + 15$
   (b) $x^2 + 4x - 12$
   (c) $x^2 - 6x - 7$
   (d) $x^3 - 3x^2 - 28x$
   (e) $x^3 - 8x^2 - 33x$
   (f) $x^3 - 2x^2 - 13x - 10$

3. For each part in this question, $p(x)$ is a different polynomial.

   (a) Let $p(x)$ be a degree 4 polynomial whose roots are $2, 3, 5, 7$. What is $p(x)$?
   (b) You are given that $p(1) = 2$ and $p(3) = 4$, and that dividing $p(x)$ by $x - 2$ leaves a remainder 0. What is $p(x)$?
   (c) The remainder when $p(x)$ is divided by $x - 2$ is 9. What is the remainder when $x \cdot p(x)$ is divided by $x - 2$?

4. This question will work through the proof of the **rational root theorem**. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial, with $a_i \in \mathbf{Z}$ for all i, and $a_n \neq 0$.

   (a) Suppose that f has a root that is a rational number $\frac{p}{q}$, assuming $\gcd(p, q) = 1$. Write the equation for the value of f at this root.
   (b) Simplify the equation from part (a) so that there are no denominators.
   (c) Isolate on one side of the equation all the terms from part (b) that contain p as a factor. What is left on the other side? What does this mean?
   (d) Isolate on one side of the equation all the terms from part (b) that contain q as a factor. What is left on the other side? What does this mean?