BBA9
BITL2

Worksheet 14

Math Lab
Fall 2020

10 December 2020

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1. **Warm up:** Answer the following True / False questions.

   (a) If $f(-1) = -1$ and $f(1) = 1$, then there is some $x \in (0, 1)$ such that $f(x) = 0$.

   (b) For every $n \in \mathbf{N}$ there is some f such that $f(1) = f(2) = f(3) = \cdots = f(n) = 0$.

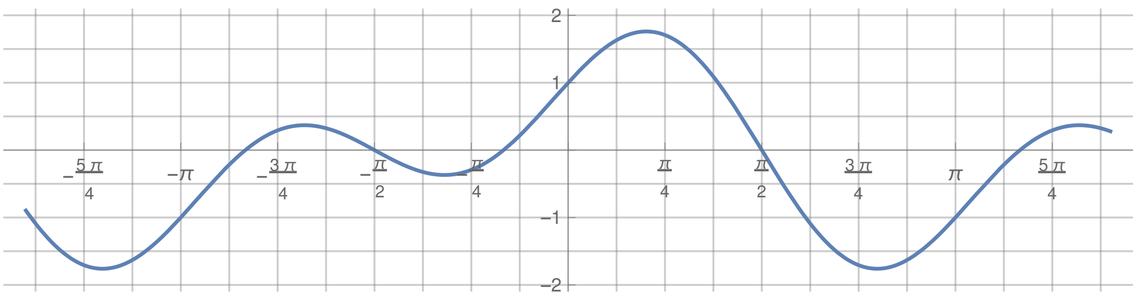
   (c) Every polynomial function is differentiable.

   (d) Every differentiable function is continuous.

2. **Newton–Raphson method:** Recall that a root of a function f is approximated iteratively by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, with x_0 the first guess and $x_{n \gg 0}$ a close approximation.

   (a) Give an example of a polynomial f and a guess x_0 for which $f(x_0) \neq 0$ but $f(x_1) = 0$.

   (b) Below is the graph of $f(x) = \sin(2x) + \cos(x)$. Using a ruler, find 10 different points on f (and not on the x-axis) so that the tangent line passes through a zero of f.



   (c) Below is the graph of $g(x) = \sin(2\pi^2/x)$. Using a calculator, compute $x_1, \ldots, x_5$ for $x_0 \in \{2.5, 2.49, 2.48, 2.47\}$. How similar is x_5 among the different choices of x_0?

3. **Infinite sequences:** For each of the following sequences $\{a_n\}_{n=1}^{\infty}$, identify which ones have a limit and which ones do not. For those that have a limit, compute it.

   (a) $a_n = \dfrac{n}{n+1}$

   (b) $a_n = \dfrac{n^2+1}{n-1}$

   (c) $a_n = \dfrac{\cos(n\pi)}{2n+1}$

   (d) $a_n = 2^{2n}$

   (e) $a_n = \dfrac{n!}{(n-1)!}$

   (f) $a_n = \left(\dfrac{-5}{6}\right)^n$

4. **Critical points and extrema:**

   (a) Find the global maximum value and the global minimum value of the function $f(x) = x^4 - 2x^2 + 3$ on the interval $[-1, 2]$.

   (b) Let $g(x) = x^3 - 3x$.

      i. Find the intervals on which $g(x)$ is increasing and on which it is decreasing.

      ii. Find the intervals on which $g(x)$ is concave up and on which it is concave down. Does f have any points of inflection?

   (c) Let $h(x) = e^{x^3-3x}$.

      i. Find the intervals on which $h(x)$ is increasing and on which it is decreasing.

      ii. Find the absolute maximum and minimum of $h(x)$ on the interval $[0, 2]$.

   (d) Consider the function $k(x) = 2x^3 + 3x^2 - 12x + 1$.

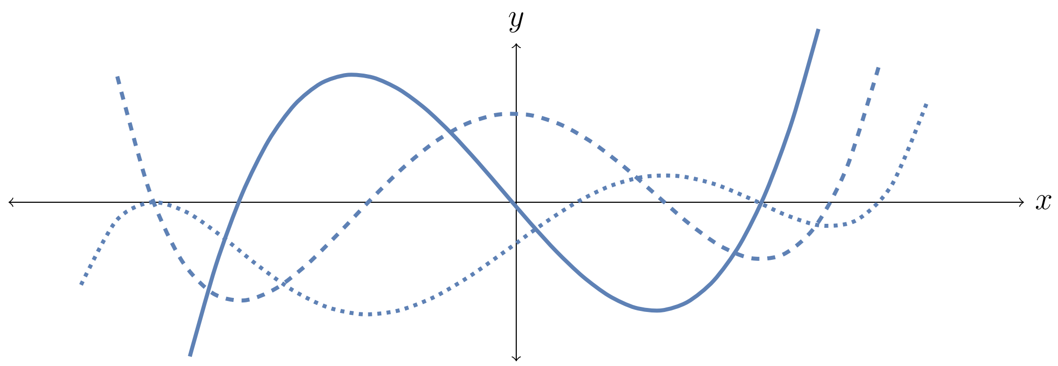
      i. Locate the critical points of k.

      ii. Use the first derivative test to locate the local maxima and local minima of k.

      iii. Use the second derivative test to locate the points of inflection, and compare your answers with part (b).

      iv. Identify the absolute minimum and maximum values of k on the interval $[-2, 4]$.

5. **Higher derivatives:** The following figure shows the graphs of f, f', and f''. Which curve is which? Justify your answers.



6. **Graph investigation:** Let $f(x) = \frac{10x^3}{x^2-1}$.

   (a) Calculate $f'(x)$ and $f''(x)$.

   (b) Does f have any vertical or horizontal asymptotes? If yes, where are they?

   (c) On what intervals is f increasing or decreasing?

   (d) At what values of x does f have a local maximum or minimum?

   (e) On what intervals is f concave upward or downward?

   (f) Sketch a graph of f.

7. **Applications of derivatives:** A base jumper dives off a cliff that is 200 meters high.

   (a) Assume the acceleration due to gravity is 9.8 meters per second squared. What is her velocity function $v(t)$, where t is time in seconds?

   (b) What is her height function $h(t)$?

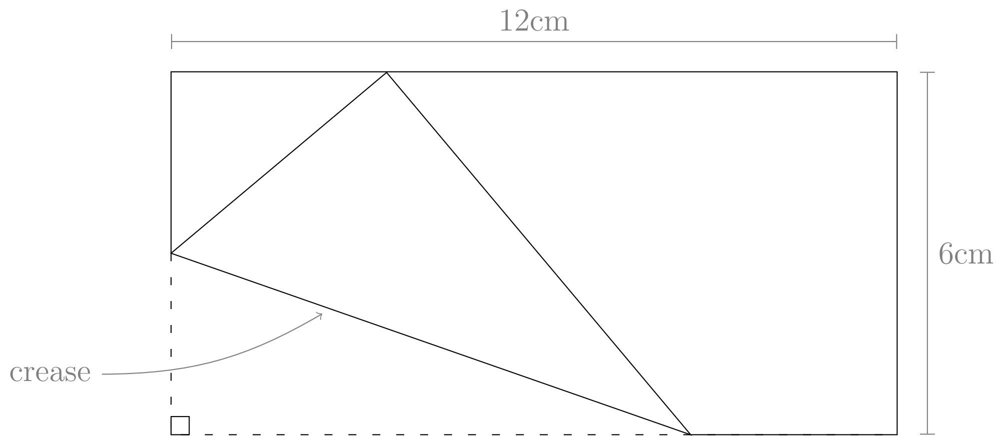
   (c) She pulls her chute when she is 100 meters down. What is her speed in this instant?

8. **Implicit differentiation:** A triangle has side lengths a, b, c which vary with time t, but its area never changes. Let θ be the angle opposite the side of length a, and suppose $\theta \in [0, \pi/2]$ for all time t.

   (a) Express $\dfrac{d\theta}{dt}$ in terms of $b, c, \theta, \dfrac{db}{dt}$ and $\dfrac{dc}{dt}$.

   (b) Express $\dfrac{da}{dt}$ in terms of the same quantities as above.

9. **Optimization:** A rectangular piece of paper is 12 cm wide and 6 cm high. The lower right-hand corner is folded over to reach the left edge of the paper, as in the picture below. What is the smallest possible length for the crease?



10. **Computing limits:** Compute the following limits or state they do not exist.

   (a) $\displaystyle\lim_{x \to +\infty} \frac{3x^5 - x^3 + 8x}{-5x^5 - 7}$

   (b) $\displaystyle\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1}$

   (c) $\displaystyle\lim_{x \to 2^-} \frac{|x - 2|}{x - 2}$

   (d) $\displaystyle\lim_{x \to -\infty} \frac{x^4 - x^2}{e^{2x}}$