BBA9
BITL2

Worksheet 3

Math Lab
Fall 2020

24 September 2020

The binomial coefficient. A binomial is a polynomial with two terms, such as $ax + b$. The coefficient of x^k in the expansion of $(ax + b)^n$, for $0 < k \leqslant n$, is $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ times $a^k b^{n-k}$.

Principle of Mathematical Induction. If $S \subset \mathbf{N}$ is a set for which

- $1 \in S$, and
- if $n \in S$, then $n + 1 \in S$,

then $S = \mathbf{N}$.

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1. **Warm up:** Factor the following polynomials using the integer root theorem.

   (a) $x^2 - 8x + 15$
   (b) $x^2 + 4x - 12$
   (c) $x^2 - 6x - 7$
   (d) $x^3 - 3x^2 - 28x$
   (e) $x^3 - 8x^2 - 33x$
   (f) $x^3 - 2x^2 - 13x - 10$

2. Using the definition of $\binom{n}{k}$ above, for $0 < k \leqslant n$, prove the following statements.

   (a) $\binom{n}{k} = \binom{n}{n-k}$
   (b) $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$
   (c) Prove by induction on n that $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$. Use the identity $\displaystyle\sum_{k=1}^{n} \binom{n}{k} = 2^n$.

3. Prove the following statements by induction.

   (a) $2^{2n-1} + 3^{2n-1}$ is divisible by 5 for all $n \in \mathbf{N}$.
   (b) $n! > 2^n$ for all $n \in \mathbf{N}_{\geqslant 4}$.
   (c) $(1 + x)^n \geqslant 1 + nx$ for all $n \in \mathbf{N}$ and $x \in \mathbf{R}_{>-1}$.

4. Consider the sums $A = \displaystyle\sum_{n=1}^{\infty} n$ and $B = \displaystyle\sum_{n=1}^{\infty} -n$.

   (a) What does $A + B$ equal?
   (b) What does $A - B$ equal?
   (c) For $C = \displaystyle\sum_{n=1}^{\infty} 1 - n$, what does $A + C$ equal?