BBA9
BITL2

Worksheet 2

Math Lab
Fall 2020

17 September 2020

In this worksheet we will use the following definitions.

- A **rational number** is a real number of the form $\frac{p}{q}$, where $p \in \mathbf{Z}$ and $q \in \mathbf{Z}_{\neq 0}$.
- A **divisor**, or **factor** of $p \in \mathbf{Z}$ is a number $d \in \mathbf{Z}$ such that dividing p by d gives a remainder of 0. That is, there is some $q \in \mathbf{Z}$ such that $p = dq$.
- A number $p \in \mathbf{Z}$ is **prime** if its only positive divisors are 1 and itself.
- The **greatest common denominator** of $p, q \in \mathbf{Z}$ is the largest number that divides both p and q. That is, it is their largest common divisor. This is written $\gcd(p, q)$.

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1. **Warm up:** Indicate which of the following functions are polynomials and which are not.

$$f(x) = \frac{1}{2}x^2 + \pi x + 2$$

$$g(x) = 3x^2 - x^{2/3}$$

$$h(x) = x^2 + 2^x$$

$$k(x) = 5x^2 + 4x + 3 + 2x^{-1}$$

$$\ell(x) = 0$$

$$m(x) = \sum_{n=1}^{1000} nx^n$$

$$p(x) = \sum_{n=-1000}^{1000} nx^n$$

$$q(x) = \sum_{n=1000}^{\infty} nx^n$$

2. Recall that saying $|x| \leqslant y$ is the same as saying $-y \leqslant x \leqslant y$.
   (a) Show that $|ab| \leqslant \frac{1}{2}(a^2 + b^2)$ for all $a, b \in \mathbf{R}$.
   (b) Show that $|a + b| \leqslant |a| + |b|$ for all $a, b \in \mathbf{R}$.

3. (a) What are the only possible integer roots of $10x^{10} - 3x^5 + 17$?
   (b) Use polynomial division to simplify the expression $\dfrac{10x^3 - 3x + 15}{3x - 2}$.
   (c) Find the roots of $x^4 + 2x^3 - 25x^2 - 26x + 120$ with polynomial division. You may assume that the roots are integers in the set $\{-5, \ldots, 5\}$,

4. Recall that the set $\mathbf{Z}$ has no largest element.
   (a) Use this to show that for any $c \in \mathbf{Q}_{>0}$, there exists $d \in \mathbf{Z}_{>0}$ with $0 < \frac{1}{d} < c$.
   (b) Use this to show that for any $c, e \in \mathbf{Q}$ with $c < e$, there exists $d \in \mathbf{Q}$ with $c < d < e$.

5. This question will work through the proof of the **rational root theorem**. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial, with $a_i \in \mathbf{Z}$ for all i, and $a_n \neq 0$.
   (a) Suppose that f has a root that is a rational number $\frac{p}{q}$, assuming $\gcd(p, q) = 1$. Write the equation for the value of f at this root.
   (b) Simplify the equation from part (a) so that there are no denominators.
   (c) Isolate on one side of the equation all the terms from part (b) that contain p as a factor. What is left on the other side? What does this mean?
   (d) Isolate on one side of the equation all the terms from part (b) that contain q as a factor. What is left on the other side? What does this mean?