

**1. Binary relations.**

- (a) Check if each of the following relations on  $\mathbf{Z}$  are reflexive, symmetric, or transitive.
- $a$  is related to  $b$  if  $\gcd(a, b) = 1$
  - $a$  is related to  $b$  if  $a \mid b$
  - $a$  is related to  $b$  if  $a$  is even
- (b) Let  $a, b \in A = \{0, 1, 2, \dots, 10\}$ . How many equivalence classes are there if:
- $a$  is related to  $b$  if  $a \equiv b \pmod{2}$ ?
  - $a$  is related to  $b$  if  $a \equiv b \pmod{5}$ ?
  - $a$  is related to  $b$  if  $a \equiv b \pmod{3}$ ?

**2. Modular arithmetic.**

- (a) What does  $a \mid b$  mean, for  $a, b \in \mathbf{Z}$ ? Write an equation.
- (b) Suppose that  $\gcd(a, b) = \gcd(b, c) = 1$ . If  $a \mid bc$  and  $c \mid ab$ , prove that  $a = c$ .
- (c) Write a general solution for all the  $n \in \mathbf{Z}$  that satisfy  $3n \equiv 5 \pmod{7}$  for all  $n$ .
- (d) Prove that every prime  $p \neq 2, 5$  divides the  $(p - 1)$ th term of  $\{9, 99, 999, 9999, \dots\}$ .

**3. Induction.**

- (a) State the Theorem of Induction.
- (b) Find a formula for  $\frac{d^n}{dx^n} x e^x$  and prove it using induction.
- (c) An island has 100 inhabitants, each having either brown or blue eyes. Due to their religion, no one knows their own eye color and no one can tell someone else theirs. If anyone determines their eye color they must leave the island forever the next day. One day a blue-eyed outsider arrives at the island. He says during a meeting with all 100 inhabitants, "it's interesting to see someone else with blue eyes so far from my homeland." What happens? Does anyone leave the island? And if so, on what day?

*Hint: Consider what happens if there are 1, 2 and 3 people with blue eyes. Use induction!*

**4. Countability.**

- (a) What does it mean for a set to be
- finite*?
  - denumerable*?
  - countable*?
- (b) Let  $A, B$  be countable sets. Prove that  $A \cup B$  is a countable set.
- (c) Prove the map  $f: \mathbf{Z} \rightarrow \mathbf{Z}$  given by  $f(n) = 2n$  is injective, but not surjective.