ESP Math 294

Worksheet 13

Fall 2018

Week of 26 November 2018

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1. Use English sentences and mathematical statements to describe the following expressions.

   (a) $a \equiv b \pmod{n}$ (b) $\gcd(a, b) = 1$ (c) $a \mid b$

Rational root theorem: *Suppose* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *is a polynomial with* $a_i \in \mathbf{Z}$. *If* $r = \frac{p}{q}$ *is a root of* f *with* $\gcd(p, q) = 1$, *then* $p \mid a_0$ *and* $q \mid a_n$.

2. (a) You may assume the following two facts:
   - Suppose that $\gcd(a, b) = 1$. Then $\gcd(a^m, b^n) = 1$, for any $m, n \in \mathbf{N}$.
   - Suppose that $\gcd(a, b) = 1$ and $a \mid bc$. Then $a \mid c$.

   Use these facts and the equation $f(r) = 0$ to write a complete proof of the rational root theorem.

   (b) Swap proofs with another group and answer the following questions for their proof.
   - Do the sentences and words make grammatical sense?
   - Is the proof easy to follow? Are there sudden jumps that leave you lost?
   - Have all the steps been correctly justified? Are any assumptions made with no justification or an implicit justification?
   - Is the proof correct? Are you convinced by the arguments?
   - Suggest changes and improvements to make the proof and its readability better.

3. Read the following proof and answer the same questions as above.

   Theorem: Everyone has the same hair color.

   Proof: This will be done by induction. For the base case, consider the set of a single person $\{P_1\}$. Clearly everyone in the set has the same hair color. Now suppose that everyone in any set $\{P_1, \ldots, P_n\}$ has the same hair color, for $n \geqslant 1$. For any other person P_{n+1}, the set $\{P_2, \ldots, P_{n+1}\}$ also has size n, so by the inductive hypothesis P_{n+1} must have the same hair color as every one of $\{P_2, \ldots, P_n\}$. Since P_1 has the same hair color as everyone in $\{P_2, \ldots, P_n\}$, everyone in $\{P_1, \ldots, P_{n+1}\}$ has the same hair color. Hence the inductive step holds, and everyone in the set of all people $\{P_1, \ldots, P_{7{,}600{,}000{,}000}\}$ has the same hair color. $\square$

4. Read the following proof and answer the same questions as above.

Theorem: The improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges only for $p > 1$.

Proof: If $p < 1$, then $\displaystyle\int_1^b \frac{1}{x^p}\,dx = \frac{b^{1-p}}{1-p} - \frac{1^{1-p}}{1-p} = \frac{b^{1-p}-1}{1-p}$
This fraction will keep increasing as $b \to \infty$. The denominator will become negligibly small, as will the term -1. Then, since $p < 1$, b^{1-p} will keep increasing to infinity. Hence for $p < 1$ the integral will not converge.
If $p = 1$, then $\displaystyle\int_1^b \frac{1}{x}\,dx = \ln(b) - \ln(1) = \ln(b)$.
As we know, the function $\ln(x)$ has no asymptote, so as $b \to \infty$, the integral for $p = 1$ will not converge.
If $p > 1$, then $\displaystyle\int_1^b \frac{1}{x^p}\,dx = \frac{-1}{(p-1)b^{p-1}} - \frac{-1}{(p-1)1^{p-1}} = \frac{b^{p-1}-1}{(p-1)b^{p-1}}$.
Now as $b \to \infty$, we will have that $b^{p-1} = b^{p-1} - 1$, so then we have

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \frac{1}{p-1}$$

Hence the integral will converge only for $p > 1$. □

5. Read the following proof and answer the same questions as above.

Theorem: Let $y = mx$ be a line through the origin that makes an angle θ with the x-axis. Let $M\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the function that reflects every point in $\mathbf{R}^2$ to its mirror image across the line. Then the matrix form of M is $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

Proof: Observing that reflecting in the line $y = x$ is equivalent to when $\theta = \frac{\pi}{4}$, the $x-$ and $y-$values get reversed, so in this situation we have the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Similarly, when reflected in the $y-$axis, we have $\theta = \frac{\pi}{2}$, and the only change is the negation of the $x-$value. In this case we have the matrix $\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Noting the change in each entry of the matrix, and observing that the difference is $\frac{\pi}{4}$ between the two, 4 functions can be easily constructed to mimic the observed nature. Hence we have: $M = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$. □