Fundamental theorem of arithmetic: For every $n \in \mathbf{Z}$ with $n > 1$, there exist unique prime numbers $p_1, \ldots, p_k \in \mathbf{N}$ and unique $e_1, \ldots, e_k \in \mathbf{N}$ such that $n = p_1^{e_1} \cdots p_k^{e_k}$.

1. Find the prime factorization of each of the following integers.

   (a) 4096

   (b) $\gcd(25!, 25^2)$

   (c) $\operatorname{lcm}(10^5, 5^{10})$

   (d) 30!

   Hint: Do not make (d) too hard for yourself - in what numbers does each prime appear?

2. Most recent books are identified by their International Standard Book Number, or ISBN, which is a 13-digit number, separated into 5 blocks. The ISBN for the Math 215 textbook is 978-0-521-59718-0. The blocks are defined as follows:

   - The first block of digits, 978, is a prefix given when the standards switched from 10 numbers to 13 numbers in 2007,

   - The second block of digits, 0, represents the language of the book (English),

   - The third block, 521, represents the publisher (Cambridge University Press),

   - The fourth block, 59718, is the number assigned to the book by the publisher,

   - The last block, 0, is the **check digit**.

   The check digit is used to determine wether an error has been made when an ISBN is copied. For an ISBN $a_1 a_2 \cdots a_{13}$ with $a_i \in \{0, \ldots, 9\}$, the check digit a_{13} is chosen so that

   $$\sum_{i=1}^{13} (2 + (-1)^i) a_i \equiv 0 \pmod{10}.$$

   (a) Determine whether 978-0-547-92822-7 is a valid ISBN.

   (b) Determine the value of the digit a so that 978-0-387-90a44-9 is a valid ISBN.

   (c) When an ISBN was copied, two adjacent digits were interchanged, resulting in the number 978-0-07-286593-4. Determine the original ISBN.

3. Prove by induction that $5^n \equiv 1 + 4n \pmod{16}$ for all $n \in \mathbf{Z}_{\geqslant 0}$.