ESP Math 294 | # Worksheet 8 | **Fall 2018**

Week of 15 October 2018

Recall the following terminology for a function:

domain, or source

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$$f\colon A \to B$$

↑ ↑

name of the function   codomain, or target

1. **Warm up:** Define the following terms, in your own words.
   (a) injective function
   (b) surjective function
   (c) image (or range) of a function

2. A set A is **infinite** if there exists a function $f : A \to A$ that is injective but not surjective.
   (a) Prove the set of natural numbers $\mathbf{N}$ is infinite.
   (b) Prove the set $\{2, 4, 6, 8\}$ is not infinite.
   (c) Prove the closed interval $[0, 1]$ is infinite.
   (d) Prove the set $\mathbf{R}$ of real numbers is infinite.

3. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function that satisfies $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbf{R}$. Prove that f is not surjective.

4. Let f, g, h be polynomials defined as

   $$f(x) = 1, \qquad g(x) = 2x + 1, \qquad h(x) = 3x^2 + 2x + 1.$$

   Let k be a polynomial of degree at most 2. Prove that there exist $r, s, t \in \mathbf{R}$ such that $k(x) = rf(x) + sg(x) + th(x)$.

5. Consider the sequence $a_n = \{9, 99, 999, 9999, 99999, \dots\}$, and let $p \neq 2, 5$ be a prime number. Prove that p divides at least one term of the sequence. *Hint: Use Fermat's little theorem.*

Binomial theorem: The coefficient of $x^a y^b$ in $(x + y)^c$, for $0 \leqslant b \leqslant a \leqslant c$, is $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

6. Prove that $\displaystyle\binom{a}{b-1} + \binom{a}{b} = \binom{a+1}{b}$.

7. Prove that $\binom{a}{b}$ is an integer for all $0 \leqslant b \leqslant a$.