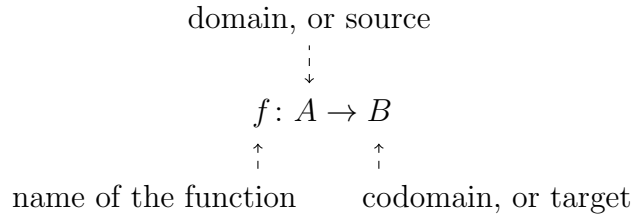


Week of 15 October 2018

Recall the following terminology for a function:



- Warm up:** Define the following terms, in your own words.
 - injective function
 - surjective function
 - image (or range) of a function
- A set A is **infinite** if there exists a function $f: A \rightarrow A$ that is injective but not surjective.
 - Prove the set of natural numbers \mathbf{N} is infinite.
 - Prove the set $\{2, 4, 6, 8\}$ is not infinite.
 - Prove the closed interval $[0, 1]$ is infinite.
 - Prove the set \mathbf{R} of real numbers is infinite.
- Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a function that satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbf{R}$. Prove that f is not surjective.
- Let f, g, h be polynomials defined as

$$f(x) = 1, \quad g(x) = 2x + 1, \quad h(x) = 3x^2 + 2x + 1.$$

Let k be a polynomial of degree at most 2. Prove that there exist $r, s, t \in \mathbf{R}$ such that $k(x) = rf(x) + sg(x) + th(x)$.

- Consider the sequence $a_n = \{9, 99, 999, 9999, 99999, \dots\}$, and let $p \neq 2, 5$ be a prime number. Prove that p divides at least one term of the sequence. *Hint: Use Fermat's little theorem.*

Binomial theorem: The coefficient of $x^a y^b$ in $(x+y)^c$, for $0 \leq b \leq a \leq c$, is $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

- Prove that $\binom{a}{b-1} + \binom{a}{b} = \binom{a+1}{b}$.
- Prove that $\binom{a}{b}$ is an integer for all $0 \leq b \leq a$.