ESP Math 294

Worksheet 5

Fall 2018

Week of 24 September 2018

Recall that $\mathbf{Q} := \{\frac{m}{n} : m \in \mathbf{Z}, n \in \mathbf{N}\}$ is the set of *rational numbers*. You may assume that:

- $\mathbf{Z}$ is closed under addition, subtraction, and multiplication.
- $\mathbf{N}$ is closed under addition and multiplication.

1. Let $a, b \in \mathbf{Q}$.

   (a) Prove that $a + b \in \mathbf{Q}$ and $a - b \in \mathbf{Q}$. That is, prove that $\mathbf{Q}$ is *closed under addition* and *closed under subtraction*.

   (b) Prove that $ab \in \mathbf{Q}$ and $\frac{a}{b} \in \mathbf{Q}$ with $b \neq 0$. That is, prove that $\mathbf{Q}$ is *closed under multiplication* and *closed under nonzero division*.

2. Recall that $\sqrt{2}$ is not a rational number, so $\sqrt{2} \notin \mathbf{Q}$. Define $\mathbf{Q}[\sqrt{2}] := \{a + b\sqrt{2} : a, b \in \mathbf{Q}\}$. The set $\mathbf{Q}[\sqrt{2}]$ is called "the rationals adjoined by the square root of 2."

   (a) Let $a, b, c, d \in \mathbf{Q}$. Prove that if $a + b\sqrt{2} = c + d\sqrt{2}$, then $a = c$ and $b = d$.

   (b) Let $x, y \in \mathbf{Q}[\sqrt{2}]$. Prove that $x + y$, $x - y$, and xy are elements of $\mathbf{Q}[\sqrt{2}]$.

   (c) Let $x, y \in \mathbf{Q}[\sqrt{2}]$ with $y \neq 0$. Prove that $\frac{x}{y}$ is an element of $\mathbf{Q}[\sqrt{2}]$.

3. Define $\mathbf{Z}[\sqrt{2}] := \{a + b\sqrt{2} : a, b \in \mathbf{Z}\}$. For $x = a + b\sqrt{2} \in \mathbf{Z}[\sqrt{2}]$, define

- the *conjugate* of x to be $\widetilde{x} := a - b\sqrt{2}$,
- the *norm* of x to be $N(x) := x\widetilde{x}$, and
- a *unit* of $\mathbf{Z}[\sqrt{2}]$ to be an element $x \in \mathbf{Z}[\sqrt{2}]$ for which $\frac{1}{x} \in \mathbf{Z}[\sqrt{2}]$.

(a) Prove that $\mathbf{Z}[\sqrt{2}] \subseteq \mathbf{Q}[\sqrt{2}]$.

(b) Let $x \in \mathbf{Z}[\sqrt{2}]$. Prove that $N(x)$ is an element of $\mathbf{Z}$.

(c) Let $x, y \in \mathbf{Z}[\sqrt{2}]$. Prove that $N(xy) = N(x)N(y)$.

(d) Let $x \in \mathbf{Z}[\sqrt{2}]$. Prove that x is a unit if and only if $N(x) = \pm 1$.

(e) Prove that $99 + 70\sqrt{2}$ is a unit.

Source: `http://www.math.uwaterloo.ca/~ajmeneze/`