**ESP Math 294** | **Fall 2018**

# Worksheet 4

Week of 17 September 2018

1. Show that all these sets have the same cardinality by defining invertible functions among them.

$$\mathbf{N} \qquad \mathbf{N} \cup \{0\} \qquad \mathbf{Z}$$

$$\{x \in \mathbf{Z} : x \text{ is divisible by } 4\} \qquad \{x \in \mathbf{Z} : x \text{ is even}\} \qquad \{3x - 10 : x \in \mathbf{N}\}$$

$$\{\tfrac{1}{a} : a \in \mathbf{N}\} \qquad \{\tfrac{a}{b} : a \in \mathbf{N}, b \in \{3,5\}\} \qquad \{(a,b) : a, b \in \mathbf{N}\}$$

2. Let $P$ be the statement: *"There exists an injective function $\mathbf{Z} \to \mathbf{Z}$ that outputs only prime numbers.* Decide which of the following statements are implied by $P$ and which imply $P$.

   (a) There are infinitely many prime numbers.

   (b) There exists an injective function $\mathbf{N} \to \mathbf{Z}$ that outputs only prime numbers.

   (c) There exists a surjective function $\mathbf{Z} \to \mathbf{Z}$ that outputs only prime numbers.

   (d) The sets $\mathbf{Z}$ and $\{x \in \mathbf{Z} : x \text{ is prime}\}$ are equal in size.

3. Use algebra to prove the following statements.

   (a) **Statement:** A right triangle with sides $a, b$ and hypotenuse $c$ (and opposing angles $A$, $B$, $C$) satisfies $a^2 + b^2 = c^2$.
   **Assumption:** You may assume the law of sines $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ for triangles.

   (b) **Statement:** For every function $f\colon \mathbf{R} \to \mathbf{R}$ there exists a real number $c_f$ such that $g\colon \mathbf{R} \to \mathbf{R}$ defined by $g(x) = f(x) + c_f$ satisfies $g(0) = 0$.

**Principle of Mathematical Induction**. If $S \subset \mathbf{N}$ is a set for which

- $1 \in S$, and
- if $n \in S$, then $n + 1 \in S$,

then $S = \mathbf{N}$.

3. Use induction to prove the following statements.

   (a) $10^n - 1$ is divisible by 3 for every $n \in \mathbf{N}$.

   (b) $\displaystyle\sum_{i=1}^{n} 3i^3 = \frac{3n^2(1+n)^2}{4}$ for every $n \in \mathbf{N}$.