ESP Math 294 | **Fall 2018**

Worksheet 3

Week of 10 September 2018

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1. Give descriptions for the following sets without the dots "...".

   (a) $\{0, 1, 2, 3, \dots\}$

   (b) $\{2, 4, 6, \dots\}$

   (c) $\{1, 3, 5, \dots\}$

   (d) $\{-10, -5, 0, 5, 10, 15, \dots\}$

   (e) $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \dots\}$

   (f) $\{0, 1, -1, 2, -2, \dots\}$

   (g) $[0, 1] \cup [2, 3] \cup [4, 5] \cup \cdots$

   (h) $[0, 1] \cap [0, 1/2] \cap [0, 1/3] \cap \cdots$

A function $f\colon A \to B$ of sets is **injective** if $a \neq a'$ in A implies $f(a) \neq f(a')$ in B. The function f is **surjective** if for every $b \in B$ there eists $a \in A$ such that $f(a) = b$.

2. For each of the functions below, decide if it is injective, surjective, both, or neither. Justify your answers.

   (a) $\begin{array}{rcl} \mathbf{Z} & \to & \mathbf{Z} \\ x & \mapsto & x^2 \end{array}$

   (b) $\begin{array}{rcl} \mathbf{N} & \to & \mathbf{N} \\ x & \mapsto & x^2 \end{array}$

   (c) $\begin{array}{rcl} \mathbf{Z} & \to & \mathbf{N} \cup \{0\} \\ x & \mapsto & |x| \end{array}$

   (d) $\begin{array}{rcl} \mathbf{Z} & \to & \mathbf{Z} \\ x & \mapsto & x + 26 \end{array}$

   (e) $\begin{array}{rcl} \mathbf{R} & \to & \mathbf{R} \\ x & \mapsto & x^3 \end{array}$

   (f) $\begin{array}{rcl} \mathbf{Q} & \to & \mathbf{Q} \\ x & \mapsto & 2x \end{array}$

   (g) $\begin{array}{rcl} \mathbf{R} & \to & \mathbf{Z} \\ x & \mapsto & \lfloor 3x \rfloor \end{array}$

   (h) $\begin{array}{rcl} \mathbf{R} & \to & \mathbf{R} \\ x & \mapsto & \sin(x) \end{array}$

3. Give three different bijective functions from $\mathbf{N}$ to $\mathbf{Z}$.

4. Give a surjective function from $\mathbf{N}$ to $\mathbf{Q} \cap (0, 1]$.

Principle of Mathematical Induction. If $S \subset \mathbf{N}$ is a set for which

- $1 \in S$, and
- if $n \in S$, then $n+1 \in S$,

then $S = \mathbf{N}$.

5. Use induction to prove the following statements.

   (a) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for every $n \in \mathbf{N}$.

   (b) $2^n + 3^n$ is divisible by 5 for each odd $n \in \mathbf{N}$.

   (c) Let $a_1 = 1$ and $a_{n+1} = \sqrt{3 + 2a_n}$ for all $n > 1$. Then $0 \leqslant a_n \leqslant a_{n+1} \leqslant 3$ for all $n \in \mathbf{N}$.

   (d) $\frac{d}{dx} x^n = n x^{n-1}$ for every $n \in \mathbf{N}$.

   (Hint: use the limit definition for $n = 1$, then the product rule.)