ESP Math 294 | **Fall 2018**

Worksheet 2

Week of 3 September 2018

This worksheet uses the following definitions:

- A **symbol** is any letter, number, shape you can think of.
- A **set** $A = \{a_1, a_2, \dots\}$ is a collection of symbols, called **elements** of the set.
- A **function** $f\colon A^n \to B$ takes as input $n \geqslant 1$ symbols $a_1, \dots, a_n$ in the set A and outputs 1 symbol $f(a_1, \dots, a_n)$ in the set B.
- The **arity** of a function is the number of symbols it has as input.

We begin by studying 1-ary, 2-ary, and 3-ary functions from the 2-element set $\{T, F\}$ to itself.

1. Make a table of values for the following binary functions. Then rewrite them using less logical connectives.

   (a) $(\neg P \implies P) \implies \neg P$

   (b) $(\neg P \iff P) \iff \neg P$

   (c) $(P \implies Q) \wedge (Q \implies P)$

   (d) $\neg P \implies \neg Q$

   (e) $\neg(P \implies \neg Q)$

   (f) $\neg Q \wedge \neg P$

2. Complete the following truth tables.

P	Q	$(P \implies Q) \iff (\neg Q \implies \neg P)$	$(P \vee Q) \vee (\neg P \vee Q)$	$(P \wedge \neg P) \vee (Q \wedge \neg Q)$
T	T			
T	F			
F	T			
F	F			

P	Q	R	$P \wedge Q \wedge R$	$(P \vee Q) \wedge R$	$P \vee (Q \wedge R)$	$P \implies (Q \vee R)$	$(P \wedge Q) \iff (Q \vee R)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Recall the basic number systems $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$. The following questions deal with the binary functions addition, subtraction, multiplication, and division, written $+, -, \times, \div$, respectively, on these sets.

3. Which descriptions of these functions are correct? Why or why not?

$+\colon \mathbf{N}^2 \to \mathbf{N}$	$+\colon \mathbf{R}^2 \to \mathbf{R}$	$+\colon \mathbf{Z}^2 \to \mathbf{R}$	$+\colon \mathbf{Q}^2 \to \mathbf{N}$
$-\colon \mathbf{N}^2 \to \mathbf{N}$	$-\colon \mathbf{Z}^2 \to \mathbf{Z}$	$-\colon \mathbf{N}^2 \to \mathbf{R}$	$-\colon \mathbf{R}^2 \to \mathbf{R}$
$\times\colon \mathbf{N}^2 \to \mathbf{N}$	$\times\colon \mathbf{Q}^2 \to \mathbf{N}$	$\div\colon \mathbf{N}^2 \to \mathbf{R}$	$\div\colon \mathbf{R}^2 \to \mathbf{R}$

4. (a) A binary function $f\colon A^2 \to B$ is **commutative** if $f(a_1, a_2) = f(a_2, a_1)$ for every a_1, a_2 in A. Which of the given binary functions are commutative? Give counterexamples for those that are not commutative.

   (b) A binary function $f\colon A^2 \to A$ is **associative** if $f(a_1, f(a_2, a_3)) = f(f(a_1, a_2), a_3)$ for every a_1, a_2, a_3 in A. Which of the given binary functions are associative? Give counterexamples for those that are not associative.

   (c) An **identity element** for a commutative binary function $f\colon A^2 \to A$ is an element e of A such that $f(e, a_1) = f(a_1, e) = a_1$ for all a_1 in A. Which of the given commutative binary functions have an identity element? What is it?

   (d) A commutative binary function $f\colon A^2 \to A$ **distributes** over another binary function $g\colon A^2 \to A$ if $f(a_1, g(a_2, a_3)) = g(f(a_1, a_2), f(a_1, a_3))$ for every a_1, a_2, a_3 in A. Which pairs of the given binary functions distribute one over the other?

5. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the unary function $f(x) = x^2$ and and $g\colon \mathbf{R}^2 \to \mathbf{R}$ the binary function $g(x, y) = (x + y)^2$.

   (a) Is g commutative?

   (b) Is the function $h\colon \mathbf{R}^2 \to \mathbf{R}$ given by $h(x, y) = g(f(x), f(y))$ commutative?

   (c) Is the function $k\colon \mathbf{R}^2 \to \mathbf{R}$ given by $k(x, y) = g(f(x) + x, f(y))$ commutative?

   (d) If we restrict the inputs of g to the set $\mathbf{Q}$, in which set do the outputs end up?

6. Give examples of:

   (a) a commutative binary function $\mathbf{Q}^2 \to \mathbf{Q}$,

   (b) a non-commutative binary function $\mathbf{R}^2 \to \mathbf{R}$,

   (c) a non-commutative binary function $\mathbf{N}^2 \to \mathbf{N}$.