ESP Math 182

Mock final - Solutions

Spring 2019

30 April 2019

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1. *Integral methods:* Evaluate the following integrals. Show all your work.

   (a) $\displaystyle\int \frac{x^2 e^{\sqrt{x^3-3}}}{\sqrt{x^3-3}}\, dx$

   For this we use integration by substitution, letting $u = \sqrt{x^3-3}$, so $du = \frac{3x^2}{\sqrt{x^3-3}}dx$ by the chain rule. That is,

   $$\int \frac{x^2 e^{\sqrt{x^3-3}}}{\sqrt{x^3-3}}\, dx = \frac{1}{3}\int e^u\, du = \frac{1}{3}\left(e^u + c\right) = \frac{e^u}{3} + c$$

   for some constant c (note that the $1/3$ is absorbed into the constant).

   (b) $\displaystyle\int x^2 \sin(2x-5)\, dx$

   This is integration by parts followed by substitution, with $u = 2x - 5$, so $du = 2dx$:

   $$\begin{aligned}\int x^2 \sin(2x-5)\, dx &= x^2 \int \sin(2x-5)\, dx - \int 2x \int \sin(2x-5)\, dx\, dx \\ &= \frac{x^2}{2}\int \sin(u)\, du - \int x \int \sin(u)\, du\, dx \\ &= \frac{x^2(-\cos(u))}{2} + \int x\cos(u)\, dx \\ &= \frac{-x^2\cos(2x-5)}{2} + \int x\cos(2x-5)\, dx,\end{aligned}$$

   and the second term is another integration by parts with the same substitution, as

   $$\begin{aligned}\int x\cos(2x-5)\, dx &= x\int \cos(2x-5)\, dx - \int\int \cos(2x-5)\, dx\, dx \\ &= \frac{x}{2}\int \cos(u)\, du - \frac{1}{2}\int\int \cos(u)\, du\, dx \\ &= \frac{x\sin(u)}{2} - \frac{1}{2}\int \sin(u)\, dx \\ &= \frac{x\sin(2x-5)}{2} - \frac{1}{2}\int \sin(2x-5)\, dx \\ &= \frac{x\sin(2x-5)}{2} - \frac{1}{4}\int \sin(u)\, du \\ &= \frac{x\sin(2x-5)}{2} + \frac{\cos(u)}{4} \\ &= \frac{x\sin(2x-5)}{2} + \frac{\cos(2x-5)}{4}\end{aligned}$$

$$= \frac{x}{2}\sin(2x-5) + \frac{1}{4}\cos(2x-5) + c.$$

(c) $\int (\csc(3x) + \cot(3x))^2 \, dx$

Expand the square and use the identity $1 + \cot^2(x) = \csc^2(x)$:

$$\begin{aligned}
(\csc(3x) + \cot(3x))^2 &= \csc^2(3x) + 2\csc(3x)\cot(3x) + \cot^2(3x) \\
&= \csc^2(3x) + 2\csc(3x)\cot(3x) + \csc^2(3x) - 1 \\
&= 2\csc^2(3x) + 2\csc(3x)\cot(3x) - 1
\end{aligned}$$

Split the integral across the three terms, use the substitution $u = 3x$, and recall the derivatives of $\cot(x)$ and $\csc(x)$, namely $\frac{d}{dx}\cot(x) = -\csc^2(x)$ and $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$:

$$\begin{aligned}
\int (\csc(3x) + \cot(3x))^2 \, dx &= \int \left(2\csc^2(3x) + 2\csc(x)\cot(3x) - 1\right) \, dx \\
&= 2\int \csc^2(3x) \, dx + 2\int \csc(3x)\cot(3x) \, dx - \int 1 \, dx \\
&= \frac{2}{3}\int \csc^2(u) \, du + \frac{2}{3}\int \csc(u)\cot(u) \, du - x \\
&= -\frac{2}{3}\cot(u) - \frac{2}{3}\csc(u) - x \\
&= -\frac{2}{3}\cot(3x) - \frac{2}{3}\csc(3x) - x.
\end{aligned}$$

(d) $\int_5^7 \frac{x+1}{9x^2+4} \, dx$

This is a trigonometric integral, which we notice by the denominator not having any real roots. We want to make this look like $1/\sqrt{u^2+a^2}$ for some u and a, and this is done by splitting it up into two terms and simplifying. Then we substitute $u = 9x^2 + 4$ (so $du = 18xdx$) in the first term, and $v = 3x$ (so $dv = 3dx$) in the second term to get

$$\begin{aligned}
\int \frac{x+1}{9x^2+4} \, dx &= \int \frac{x}{9x^2+4} \, dx + \int \frac{1}{9x^2+4} \, dx \\
&= \frac{1}{18}\int \frac{1}{u} \, du + \frac{1}{3}\int \frac{1}{v^2+2^2} \, dv \\
&= \frac{\ln(|u|)}{18} + \frac{\arctan(v/2)/2}{3} \\
&= \frac{\ln(9x^2+4)}{18} + \frac{\arctan(3x/2)}{6}.
\end{aligned}$$

Evaluate this from $x = 5$ to $x = 7$ to get the answer.

(e) $\displaystyle\int_e^3 \frac{x^2+x-20}{x^3-4x^2+4x}\,dx$

Here we have to use partial fractions. Factoring shows that

$$x^x + x - 20 = (x+5)(x-4), \qquad x^3 - 4x^2 + 4x = x(x-2)^2.$$

Hence we have some constants A, B, C such that

$$\frac{x^2+x-20}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2},$$

or

$$x^2 + x - 20 = A(x-2)^2 + Bx(x-2) + Cx.$$

Evaluating this equation at the points 0 and 2 gives that $-20 = 4A$ and $-14 = 2C$. To get B, we compare coefficients of the x^2 terms on both sides, getting $1 = A + B$. Hence $A = -5$, $B = 6$ and $C = -7$. The integral can now be distributed to the terms as follows:

$$\int_e^3 \frac{-5}{x}\,dx = -5\ln(|x|)\Big|_{x=e}^{x=3} = -5\ln(3) - 5,$$

$$\int_e^3 \frac{6}{x-2}\,dx = \int_{e-2}^1 \frac{6}{u}\,du = 6\ln(|u|)\Big|_{u=e-2}^{u=1} = -6\ln(e-2),$$

$$\int_e^3 \frac{-7}{(x-2)^2} \int_{e-2}^1 \frac{-7}{u^2}\,du = \frac{7}{u}\Big|_{u=e-2}^{u=1} = 7 - \frac{7}{e-2}.$$

(f) $\displaystyle\int_1^{2\pi} e^x\cos(x)\,dx$

This is two applications of integration by parts, then rearranging. First let $u = e^x$ and $dv = \cos(x)\,dx$, so that $du = e^x\,dx$ and $v = \sin(x)$:

$$\int e^x\cos(x)\,dx = e^x\sin(x) - \int e^x\sin(x)\,dx.$$

Use integration by parts on the second term, letting $u = e^x$ again, and $dv = \sin(x)\,dx$, so that $du = e^x\,dx$ and $v = -\cos(x)$:

$$\int e^x\sin(x)\,dx = -e^x\cos(x) - \int e^x(-\cos(x))\,dx.$$

The original integral $\int e^x\cos(x)\,dx$ has appeared, hinting that we may need to rearrange (group the terms together). This does indeed work out:

$$\int e^x\cos(x)\,dx = e^x\sin(x) - \left(-e^x\cos(x) - \int e^x(-\cos(x))\,dx\right)$$

$$\int e^x\cos(x)\,dx = e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)\,dx$$

$$2\int e^x\cos(x)\,dx = e^x\left(\sin(x) + \cos(x)\right)$$

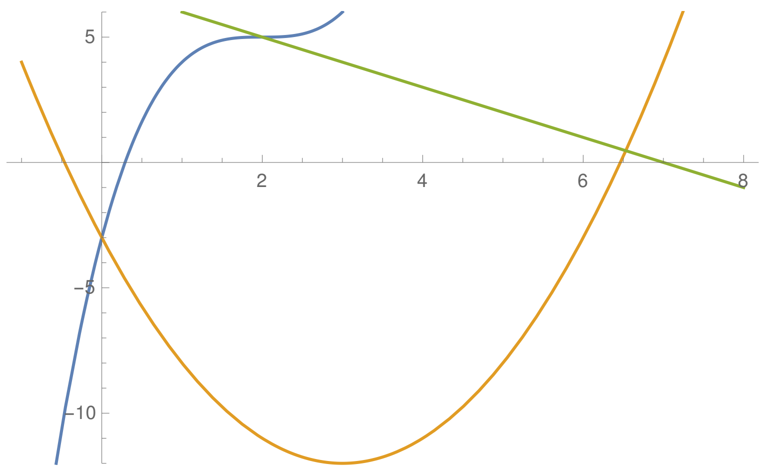
$$\int e^x \cos(x)\, dx = \frac{e^x}{2}\left(\sin(x) + \cos(x)\right).$$

This was a definite integral, so we need to evaluate it at the bounds:

$$\begin{aligned}\int_1^{2\pi} e^x \cos(x)\, dx &= \frac{e^x}{2}\left(\sin(x) + \cos(x)\right)\Big|_{x=1}^{x=2\pi} \\ &= \left(\frac{e^{2\pi}}{2}\left(\sin(2\pi) + \cos(2\pi)\right)\right) - \left(\frac{e^1}{2}\left(\sin(1) + \cos(1)\right)\right) \\ &= \frac{e^{2\pi}}{2} - \frac{e}{2}\left(\sin(1) + \cos(1)\right).\end{aligned}$$

2. *Area between curves:* Find the integral that represents the area above the curve $y = (x-3)^2 - 12$ and below both of the curves $y = (x-2)^3 + 5$ and $y = 7 - x$. Do not evaluate the integral. *Hint: The cubic and linear curves intersect at* $x = 2$.

The graphs of these three functions on the interval $[-1, 8]$ with range $[-12, 6]$ is below.



The intersection point of the parabola with the cubic function is found to be at

$$\begin{aligned}(x-3)^2 - 12 &= (x-2)^3 + 5 \\ x^2 - 6x + 9 - 12 &= x^3 - 6x^2 + 12x - 8 + 5 \\ x^3 - 7x^2 + 18x &= 0 \\ x(x^2 - 7x + 18) &= 0.\end{aligned}$$

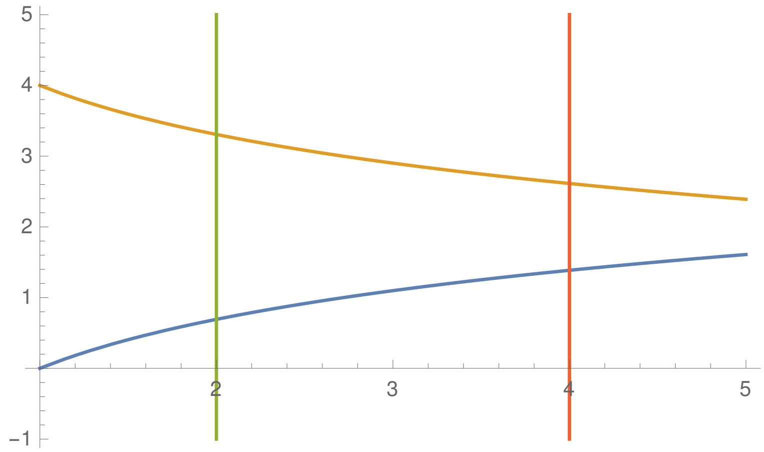
Since $(-7)^2 - 4 \cdot 18 = 49 - 72 = -23 < 0$, the discriminant of the quadratic factor is negative, so the only solution to the equation is $x = 0$, for which $y = -3$. With the hint, we find the intersection of the cubic with the line at $(2, 5)$. With the quadratic formula, we find the intersection of the quadratic with the line at $((5 + \sqrt{65})/2, (9 - \sqrt{65})/2)$. Hence the area of the shape is given by

$$\int_0^2 ((x-2)^3 + 5) - ((x-3)^2 - 12)\, dx + \int_2^{(5+\sqrt{65})/2} (7 - x) - ((x-3)^2 - 12)\, dx.$$

3. *Volumes of revolution:* Calculate the following volumes using the disk method.

(a) The area bounded by $y = \ln(x)$, $y = 4 - \ln(x)$, $x = 2$, and $x = 4$ revolved around the x-axis.

The four curves are given in the diagram below.

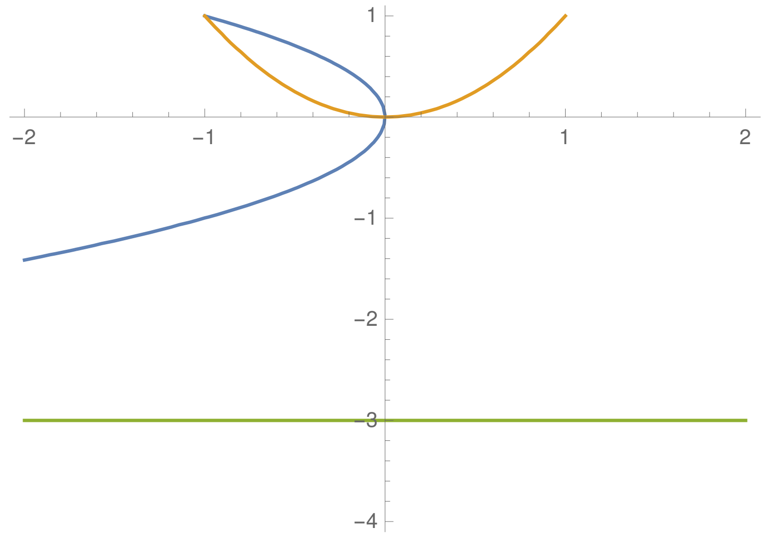


Hence the integral representing the volume is

$$\begin{aligned}\pi \int_2^4 (4-\ln(x))^2 - (\ln(x))^2 \, dx &= \pi \int_2^4 16 - 8\ln(x) \, dx \\ &= \pi \left(16x - 8(x\ln(x) - x)\right) \Big|_{x=2}^{x=4} \\ &= \pi \left((64 - 8(4\ln(4) - 4)) - (32 - 8(2\ln(2) - 2))\right) \\ &= \pi \left(64 - 32\ln(4) + 32 - 32 + 16\ln(2) - 16\right) \\ &= \pi \left(48 - 64\ln(2) + 16\ln(2)\right) \\ &= 48\pi(1 - \ln(2)).\end{aligned}$$

(b) The area in the second quadrant bounded by $x = -y^2$ and $y = x^2$ revolved around the axis $y = -3$.

The two curves and the axis $y = -3$ are given the in the diagram below.

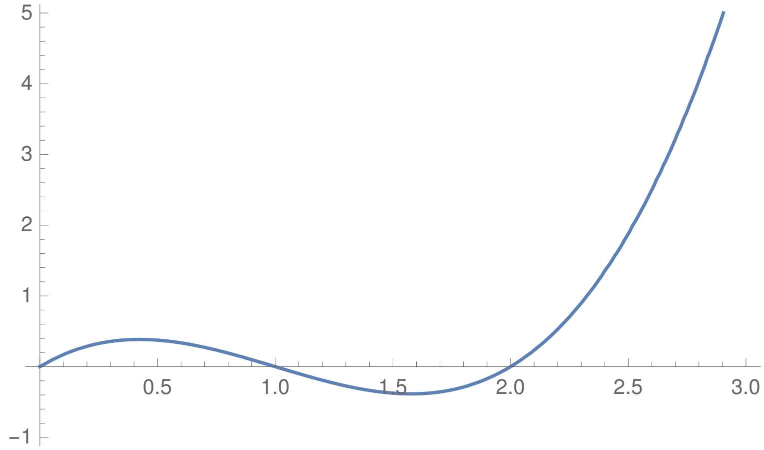


Expressing the curve $x = -y^2$ in terms of x we get $y = \pm\sqrt{-x}$. We chose the positive side $+\sqrt{-x}$, because that is the one above the x-axis. Since we are rotating not around the x-axis, but around a line shifted three units below the x-axis, we have to add 3 to both functions to get the right shape. Hence the integral representing the volume is

$$\begin{aligned}
\pi \int_{-1}^{0} (\sqrt{-x}+3)^2 - (x^2+3)^2 \; dx &= \pi \int_{-1}^{0} -x + 3\sqrt{-x} + 9 - x^4 - 6x^2 - 9 \; dx \\
&= \pi \left(-\int_{-1}^{0} x \; dx + 3\int_{-1}^{0} \sqrt{-x} \; dx - \int_{-1}^{0} x^4 \; dx - 6\int_{-1}^{0} x^2 \; dx \right) \\
&= \pi \left(\frac{x^2}{2} + 2(-x)^{3/2} - \frac{x^5}{5} - 2x^3 \right) \Bigg|_{x=-1}^{x=0} \\
&= \pi \left(\frac{(-1)^2}{2} + 2(1)^{3/2} - \frac{(-1)^5}{5} - 2(-1)^3 \right) \\
&= \pi \left(\frac{1}{2} + 2 + \frac{1}{5} + 2 \right) \\
&= \frac{47\pi}{10}.
\end{aligned}$$

(c) The volume of revolution of $y = x(x-1)(x-2)$ revolved around the x-axis between $x = 0$ and $x = 3$.

The curve is given in the diagram below.



Here we simply integrate from 1 to 3 with the height of the function as the radius of the disks. So the volume of the solid is

$$\pi \int_0^3 (x(x-1)(x-2))^2 \, dx = \frac{288\pi}{35}.$$

The calculations are skipped because the integrand is just a polynomial, with no tricks.

4. *Sequences:* For each of the following sequences, determine if it converges or diverges. If it converges find the limit.

(a) $x_n = \dfrac{n}{n+1}$

Observe that

$$\lim_{n\to\infty}\left[\frac{n}{n+1}\right] = \lim_{n\to\infty}\left[\frac{\frac{1}{n}}{\frac{1}{n}} \cdot \frac{n}{n+1}\right] = \lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}}\right] = \frac{1}{1+0} = 1,$$

so the sequence converges, and converges to 1.

(b) $x_n = \dfrac{n\cos(n\pi)}{2n+1}$

Observe that when n is an odd number, $\cos(n\pi) = \cos(\pi) = -1$, so then

$$\lim_{n\to\infty}\left[\frac{n\cos(n\pi)}{2n+1}\right] = \lim_{n\to\infty}\left[\frac{-n}{2n+1}\right] = \lim_{n\to\infty}\left[\frac{-1}{2+\frac{1}{n}}\right] = \frac{-1}{2+0} = -\frac{1}{2},$$

but if n is odd, then $\cos(n\pi) = \cos(0) = 1$, so then

$$\lim_{n\to\infty}\left[\frac{n\cos(n\pi)}{2n+1}\right] = \lim_{n\to\infty}\left[\frac{n}{2n+1}\right] = \lim_{n\to\infty}\left[\frac{1}{2+\frac{1}{n}}\right] = \frac{1}{2+0} = \frac{1}{2},$$

so the limits are not the same. That is, the sequence alternates between $1/2$ and $-1/2$ forever. Hence the sequence does not converge.

(c) $x_n = \dfrac{\sin(n)}{n}$

This is an application of the squeeze theorem. Recall that $-1 \leqslant \sin(x) \leqslant 1$ for any argument x, so then

$$\begin{aligned} -1 \leqslant \sin(n) \leqslant 1 \\ -\frac{1}{n} \leqslant \frac{\sin(n)}{n} \leqslant \frac{1}{n} \\ \lim_{n\to\infty}\left[-\frac{1}{n}\right] \leqslant \lim_{n\to\infty}\left[\frac{\sin(n)}{n}\right] \leqslant \lim_{n\to\infty}\left[\frac{1}{n}\right] \\ -0 \leqslant \lim_{n\to\infty}\left[\frac{\sin(n)}{n}\right] \leqslant 0. \end{aligned}$$

Hence the sequence $\sin(n)/n$ converges to 0.

5. *Series - convergence / divergence tests:* Determine if the following series converge or diverge. Indicate which tests you have used.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^2+1}}$

Observe that the limit of the individual terms as $n \to \infty$ does not equal 0, so by the divergence test, the series must diverge:

$$\lim_{n\to\infty}\left[\frac{2n+1}{\sqrt{n^2+1}}\right] = \lim_{n\to\infty}\left[\frac{2+\frac{1}{n}}{\sqrt{1+\frac{1}{n^2}}}\right] = 2 \neq 0.$$

(b) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{n}{12} - \frac{n+1}{6}\right)$

This looks like a telescoping series, but the limit of the individual terms has no lower bound, and so again by the series divergence test, this series diverges:

$$\lim_{n\to\infty}\left[\frac{n}{12} - \frac{n+1}{6}\right] = \lim_{n\to\infty}\left[\frac{n-2n-2}{12}\right] = \frac{1}{12}\lim_{n\to\infty}\left[-n-2\right] \to -\infty \neq 0.$$

6. *Series - sum of series:* Find the value of the following convergent series. Indicate what type of series they are.

(a) $\displaystyle\sum_{n=0}^{\infty} 2^{2n}4^{3n+1}e^{8-8n}$

Algebraic simplification of the summand shows this is a geometric series:

$$2^{2n}4^{3n+1}e^{8-8n} = \frac{\left(2^2\right)^n\left(4^3\right)^n 4^1 e^8}{\left(e^8\right)^n} = \left(\frac{2^2\cdot 4^3}{e^8}\right)^n 4e^8 = \left(\frac{2^8}{e^8}\right)^n 4e^8.$$

Since $e > 2$, the ratio $r = \frac{2^8}{e^8} < 1$, so the series does indeed converge. Here $a = 4e^8$ is the first term, so by the formula, the sum of the series is

$$\frac{a}{1-r} = \frac{4e^8}{1 - \frac{2^8}{e^8}} = \frac{4e^{16}}{e^8 - 2^8}.$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{4}{n^2 + 4n + 3}$

Since $n^2 + 4n + 3 = (n+1)(n+3)$, we can use partial fractions to split up the summand into two terms:

$$\frac{4}{n^2+4n+3} = \frac{A}{n+1} + \frac{B}{n+3}$$
$$4 = A(n+3) + B(n+1).$$

At $n = -1$, we get that $A = 2$, and at $n = -3$ we get $B = -2$. Hence the series is

$$2\sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right),$$

which is telescoping with partial sums

$$S_2 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)$$
$$S_3 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right)$$
$$S_4 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right)$$
$$\vdots$$
$$S_n = 1 + \frac{1}{2} - \frac{1}{n+2} - \frac{1}{n+3}.$$

The limit of these partial sums is $\frac{3}{2}$, and as the original series had a multiple of 2 facored out, the series converges to 3.

7. *Series - intervals of convergence:* Find the intervals of convergence of the following series. Indicate which tests you have used.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(x-2)^n}{(n\ln(n))^2}$

Note that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(x-2)^{n+1}}{((n+1)\ln(n+1))^2}}{\frac{(x-2)^n}{(n\ln(n))^2}} = \frac{(x-2)(n\ln(n))^2}{((n+1)\ln(n))^2}.$$

We only take the factors inside the square as $n \to \infty$, and apply l'Hopital's rule to get that

$$\lim_{n\to\infty}\left[\frac{n\ln(n)}{(n+1)\ln(n)}\right] = \lim_{n\to\infty}\left[\frac{\ln(n)+\frac{n}{n}}{\ln(n)+\frac{n}{n}+\frac{1}{n}}\right] = \lim_{n\to\infty}\left[\frac{1+\frac{1}{\ln(n)}}{1+\frac{1}{\ln(n)}+\frac{1}{n\ln(n)}}\right] = 1.$$

The square of the limit also goes to 1, so the series certainly converges for $|x-2|<1$, or $1<x<3$. At the endpoints, we have that

$$\frac{(3-2)^n}{(n\ln(n))^2} = \frac{1}{(n\ln(n))^2} \leqslant \frac{1}{n^2},$$

which converges by the p-series test. The other endpoint converges by the alternating series test, so the interval of convergence for this series is $x \in [1,3]$.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{15^n n}$

Note that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(x-3)^{n+1}}{15^{n+1}(n+1)}}{\frac{(x-3)^n}{15^n n}} = \frac{(x-3)n}{15(n+1)},$$

and taking the limit of this as $n\to\infty$, we get

$$\lim_{n\to\infty}\left[\frac{(x-3)n}{15(n+1)}\right] = \frac{x-3}{15}\lim_{n\to\infty}\left[\frac{n}{n+1}\right] = \frac{x-3}{15}\lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}}\right] = \frac{x-3}{15},$$

so by the ratio test, we have that the series definitely converges for $|\frac{x-3}{15}|<1$, or $-12<x<18$. For the endpoints, we have that

$$\frac{(18-3)^n}{15^n n} = \frac{1}{n}, \qquad \frac{(-12-3)^n}{15^n n} = \frac{(-1)^n}{n},$$

both of which are divergent series. Hence the interval of convergence is $x\in(-12,18)$ for this series.

8. *Power series:*

(a) Find the first four terms of the Maclaurin series of $f(x) = \int_0^x 3t^3 - \frac{5}{2}t^2 + 2\,dt$.

The first term is $f(0)$, which, being the integral from 0 to 0, is $f(0)=0$. The second term is the derivative of f at 0, so by the fundamental theorem of calculus, we have

$$f'(0) = 3(0)^3 - \frac{5}{2}(0)^2 + 2 = 2.$$

The derivative is $f'(x) = 3x^3 - \frac{5}{2}x^2 + 2$. The third term needs the second derivative, which is $f''(x) = 9x^2 - 5x$. Hence the thrid term is

$$\frac{f''(0)}{2} = \frac{9(0)^2 - 5(0)}{2} = 0.$$

The fourth term needs the third derivative, which is $f'''(x) = 18x - 5$. Hence the fourth term is

$$\frac{f'''(0)}{6} = \frac{18(0) - 5}{6} = -\frac{5}{6}.$$

Hence the first four terms of the Maclaurin series are $0 + 2 + 0 - \frac{5}{6}$.

(b) What are the Maclaurin series of the following common functions?

i. e^x

Either by direct computation or by recognizing the series, this is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

ii. $\frac{1}{1+x}$

Either by direct computation or by recognizing the series, this is $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$.

iii. $\cos(x)$

Either by direct computation or by recognizing the series, this is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

(c) Find the first three terms of the Taylor series of $f(x) = 2e^{2x}\sin(2x)$ at $x = a$.

For a function f, recall the first three terms of the Taylor series of f at a are

$$f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2}.$$

We get the first term easily to be $f(a) = 2e^{2a}\sin(2a)$. To get the second term, we differentiate, and use the product rule:

$$f'(x) = 4e^{2x}\sin(2x) + 4e^{2x}\cos(2x).$$

Hence the second term of the Taylor series is $4e^{2a}(\sin(2a) + \cos(2a))(x-a)$. To get the third and final term, we need to differentiate again:

$$\begin{aligned} f''(x) &= 8e^{2x}\sin(2x) + 8e^{2x}\cos(2x) + 8e^{2x}\cos(2x) - 8e^{2x}\sin(2x) \\ &= 16e^{2x}\cos(2x). \end{aligned}$$

It is immediate that the third term is $16e^{2a}\cos(2a)\frac{(x-a)^2}{2}$.

9. *Parametric equations:*

(a) Describe the linear system

$$\begin{aligned} 4x + 5y - 2z &= 7, \\ x - y + 10z &= 1 \end{aligned}$$

as a parametric equation in the variable t.

We choose z to be our free variable (but any other would work). Solve the second equation for x (as that is easier) and replace it in the first to get

$$4(1 + y - 10z) + 5y - 2z = 7 \qquad \Longrightarrow \qquad y = \frac{42z + 3}{9}.$$

Set $t = z$ and replace this in the second equation to get

$$x = 1 + \frac{42t+3}{9} + 10t = \frac{132t+12}{9}.$$

Hence the parametric equation describing this linear system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 132/9 \\ 42/9 \\ 1 \end{pmatrix} t + \begin{pmatrix} 4/3 \\ 1/3 \\ 0 \end{pmatrix}.$$

(b) For the parametric curve $(x, y) = (5t - 2, 8 - 3t)$, find $\frac{dy}{dx}$ and the values of t for which the graph is in the first quadrant.

Recall that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3}{5}.$$

For the first quadrant, we must have both the x-values and y-values be positive, so

$$\begin{aligned} 5 - 2t &\geqslant 0 \\ 5t &\geqslant 2 \\ t &\geqslant 2/5, \end{aligned} \qquad \begin{aligned} 8 - 3t &\geqslant 0 \\ 8 &\geqslant 3t \\ 8/3 &\geqslant t. \end{aligned}$$

In other words, we must have $2/5 \leqslant t \leqslant 8/3$ for the graph to be in the first quadrant.

10. *Matrices:* Find the determinant, eigenvalues, and eigenvectors of the matrix $\left[\begin{smallmatrix} 1 & 1 \\ -1 & 2 \end{smallmatrix}\right]$.

The determinant is $1 \cdot 2 - 1 \cdot (-1) = 3$. Recall the eigenvalues of a matrix A are the roots of the polynomial $\det(A - \lambda x) = 0$. That is,

$$\begin{aligned} 0 &= \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) + 1 = \lambda^2 - 3\lambda + 3, \\ \lambda &= \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{3 \pm \sqrt{-3}}{2}. \end{aligned}$$

Both of these eigenvalue are imaginary, that is, they don't exist, as we cannot take the square root of a negative number. Hence there are no eigenvalues and so there are no eigenvectors.