**ESP Math 182** 

# Worksheet 22

**Spring 2019**

23 April 2019

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<u>Determinants:</u> Only for square matrices. A number that is 0 if the matrix has no inverse, and nonzero otherwise. The general formula is complicated, but for small matrices we have

$$\det \begin{bmatrix} a \end{bmatrix} = a, \quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc, \quad \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

<u>Eigenvalues:</u> An *eigenvalue* of a matrix $A$ is a non-zero number $\lambda$ such that $A\vec{x} = \lambda\vec{x}$, for some vector $\vec{x}$, called the *eigenvector* of $\lambda$. Be careful - not all matrices have eigenvalues or eigenvectors!

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1. **Warm up:** Find the determinants of the following matrices.

   (a) $\begin{bmatrix} 5 & 3/4 \\ -2 & 7/3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -11 & 8 \\ 0 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 1 \end{bmatrix}$

2. The *rotation matrix* $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ rotates any 2-vectors by an angle of $\theta$.

   (a) What is the determinant of the rotation matrix?
   (b) Calculate $R^2$.
   (c) Find a matrix $S$ such that $S^2 = R$ (the square root of $R$). *Hint: think geometrically.*

3. For each function $f$, find values $a$ such that $f(a) = a$.

   (a) $f(x) = e^x - 1$ (b) $f(x, y) = (x, 2y - 2)$ (c) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

4. Find the eigenvalues and associated eigenvectors and of the following linear maps.

   (a) $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

5. For the matrices in the previous question:

   (a) Draw where the vectors $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$ get taken to and color in the shape (called a *parallelogram*) they bound.
   (b) Find the area of the shape.
   (c) Compare the area of the shape with the determinant of the matrix.

6. Consider a function $\mathbf{R}^2 \to \mathbf{R}$ defined by $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \mapsto \det\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} x & 1 \\ 1 & y \end{smallmatrix}\right]\right)$.

   (a) Take the derivative with respect to $x$, then with respect to $y$.
   (b) For what vectors $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is the value of the function equal to $ad - bc$?
   (c) Evaluate the derivatives at these values.