ESP Math 182

Worksheet 21

Spring 2019

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<u>LINEAR SYSTEMS:</u> Recall that $\mathbf{R}^n = \{(v_1, \ldots, v_n) \ : \ v_i \in \mathbf{R}\}$ is a *vector space*, with *basis* elements

$$e_1 = (1, 0, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \quad \ldots, \quad e_n = (0, \ldots, 0, 0, 1).$$

A *linear equation* in $\mathbf{R}^n$ is a linear polynomial $a_0 + a_1x_1 + \cdots + a_nx_n = 0$, where $a_i \in \mathbf{R}$ and the x_i are *indeterminates*, or *variables*. A linear combination of elements $c_1e_1 + \cdots + c_ne_n$, for $c_i \in \mathbf{R}$, is a *solution* to this equation if $a_0 + a_1c_1 + \cdots + a_nc_n = 0$. A *linear system* is a collection

$$\begin{aligned} a_{1,0} + a_{1,1}x_1 + \cdots + a_{1,n}x_n &= 0, \\ a_{2,0} + a_{2,1}x_1 + \cdots + a_{2,n}x_n &= 0, \\ &\vdots \\ a_{k,0} + a_{k,1}x_1 + \cdots + a_{k,n}x_n &= 0 \end{aligned}$$

of linear equations. A linear combination of elements $c_1e_1 + \cdots + c_ne_n$ is a *solution* to this system if $a_{i,0} + a_{i,1}c_1 + \cdots + a_{i,n}c_n = 0$ for all $i = 1, \ldots, k$. The *solution space* is the collection of elements of $\mathbf{R}^n$ that satisfy all the equations in a system, itself a vector space.

A linear system has *non-degenerate* equations, which are equations that have solutions by themselves in the given vector space. For example, $x_1 - 2 - x_1 = 0$ is degenerate because it simplifies to $1 = 0$. We eliminate *dependent* equations in a linear system, which are linear combinations of the other equations. Every *independent* equation reduces the solution space by 1 dimension.

The vector space $\mathbf{R}^n$ with an empty linear system has a dimension n solution space, as all variables x_i are *independent*, or *free*. Every independent equation in the system makes one of the independent variables *dependent*, though you have a choice as to which becomes dependent.

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1. For each of the following systems of linear equations, find at least one solution in the appropriate vector spaces. If no solutions exist, say so.

   (a) $5 + 4x_1 = 0$ in $\mathbf{R}^2$

   (b) $1 + 2x_1 = 0$ in $\mathbf{R}^1$
   $1 + 3x_1 = 0$

   (c) $1 + 2x_1 = 0$ in $\mathbf{R}^2$
   $1 + 3x_1 = 0$

   (d) $2 + 2x_1 = 0$ in $\mathbf{R}^4$
   $1 - x_2 = 0$
   $-2 + \pi x_3 = 0$

2. For each of the following linear systems, find all the solutions in the solutions space, an independent collection of equations, and indicate the dimension of the solution space.

   (a) $$\begin{aligned} -8x_2 &= 0 \quad \text{in } \mathbf{R}^5 \\ 2 + x_1 + 9x_2 &= 0 \\ x_2 - 2x_5 + 3x_1 + 2 &= 0 \end{aligned}$$

   (b) $$\begin{aligned} x_1 + 3x_2 - x_4 &= 0 \quad \text{in } \mathbf{R}^4 \\ x_3 - x_1/2 + 4 &= x_3 - 7 \\ 3x_2 + 22 &= x_4 \end{aligned}$$

Matrices: Recall that an $m \times n$ *matrix* A is a collection of mn elements, represented by A_{ij} for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

- The *sum* of two $m \times n$ matrices A, B is an $m \times n$ matrix $C_{ij} = A_{ij} + B_{ij}$
- The *product* of an $m \times n$ matrix A and an $n \times r$ matrix B is a $m \times r$ matrix $C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$
- The *transpose* of an $m \times n$ matrix A is an $n \times m$ matrix $(A^T)_{ij} = A_{ji}$
- The *inverse* of an $n \times n$ matrix A is an $n \times n$ matrix A^{-1} such that $AA^{-1} = I_n$
- To a matrix A we may apply *elementary row operations* to its rows R_k:
  - $R_k \to cR_k$, for $c \neq 0$
  - $R_k \to R_\ell$ and $R_\ell \to R_k$
  - $cR_k + R_\ell \to R_\ell$
- The *reduced echelon form* of an $m \times n$ matrix A is the $m \times n$ matrix $R = [I_m|B]$, for B an $m \times (n-m)$ matrix, and R obtained from A by elementary row operations and column swapping. We assume $n > m$.

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3. Find coefficient vectors $\vec{x}$ that make the following equalities true.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

4. Turn the linear systems, in the given vector space, into augmented matrices.

(a) $$\begin{aligned} x_1 - 3x_2 &= 5 \quad \text{in } \mathbf{R}^3 \\ 9 - x_2 - x_3 &= x_1 + 2 \end{aligned}$$

(b) $$\begin{aligned} 2 + 2x_1 &= x_4 \quad \text{in } \mathbf{R}^4 \\ 1 - x_2 + 5x_1 &= 7 \\ -2 + \pi x_4 &= 0 \end{aligned}$$

5. By elementary row operations, bring the following matrix to a reduced echelon matrix (that is, make it look like the matrix on the right). Show the row operations that you carry out.

$$\begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix}$$

6. This question will explore properties of the space of all matrices.

(a) Find two 2×2 matrices A, B such that $AB \neq BA$. This means matrices are not *commutative*.

(b) Find two 2×2 matrices C, D such that $CD = 0$, but $C \neq 0$ and $D \neq 0$. This means matrices are not an *integral domain*.

(c) Show that for all 2×2 matrices A, B, C, we have $(AB)C = A(BC)$. This means that matrices are *associative*.