ESP Math 182

Worksheet 15

Spring 2019

7 March 2019

---

1. **Warm up:** The following series are all convergent. Indicate which test applies to determine convergence.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{10}{9^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi/3}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

2. Use l'Hopital's rule to evaluate the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

   (b) $\displaystyle\lim_{x\to 0^+} x^{\sin(x)}$

   (c) $\displaystyle\lim_{y\to 0} \frac{\tan(y) - y}{y^3}$

   (d) $\displaystyle\lim_{z\to\infty} ze^{1/z} - z$

3. Recall the integral test says that for a non-increasing function f, the sum $\sum_{n=N}^{\infty} f(n)$ converges if and only if the integral $\int_N^{\infty} f(x)\, dx$ is finite.

   (a) Show that $\displaystyle\sum_{n=2}^{\infty} \left(\frac{e}{n}\right)^n$ converges.

   (b) Show that $\displaystyle\int_1^{\infty} \frac{e^y}{y^y}\, dy$ converges.

   (c) Determine whether or not $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$ converges.

4. You are given the two inequalities

$$x - \frac{x^2}{2} < \ln(1+x) < x, \qquad \text{for } x > 0, \tag{1}$$

$$\ln(1) + \ln(2) + \cdots + \ln(n-1) < \int_1^n \ln(x)\,dx < \ln(2) + \cdots + \ln(n), \qquad \text{for } n \geqslant 2. \tag{2}$$

Use them to answer the following questions.

(a) Use inequality (1) to prove that $\lim_{x\to 0^+} \left[\frac{\ln(1+x)}{x}\right] = 1.$

(b) Use part (a) to prove that $\lim_{n\to\infty} \left[\left(1 + \frac{1}{n}\right)^n\right] = e.$

(c) Show that $\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}$ converges for $0 < a < e$ and diverges for $a > e$.

(d) Use inequality (2) to show that $\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}.$

(e) Use part (d) to determine if the series from part (c) converges if $a = e$.