ESP Math 182

Worksheet 26

Spring 2018

19 April 2018

Recall that $\mathbf{R}^n = \{(v_1, \ldots, v_n) \ : \ v_i \in \mathbf{R}\}$ is a *vector space*, with *basis* elements

$$\begin{aligned} e_1 &= (1,0,0,\ldots,0), \\ e_2 &= (0,1,0,\ldots,0), \\ &\vdots \\ e_{n-1} &= (0,\ldots,0,1,0), \\ e_n &= (0,\ldots,0,0,1). \end{aligned}$$

A *linear equation* in $\mathbf{R}^n$ is a linear polynomial $a_0 + a_1x_1 + \cdots + a_nx_n = 0$, where $a_i \in \mathbf{R}$ and the x_i are *indeterminates*, or *variables*. A linear combination of elements $c_1e_1 + \cdots + c_ne_n$, for $c_i \in \mathbf{R}$, is a *solution* to this equation if $a_0 + a_1c_1 + \cdots + a_nc_n = 0$. A *system* of linear equations is a collection

$$\begin{aligned} a_{1,0} + a_{1,1}x_1 + \cdots + a_{1,n}x_n &= 0, \\ a_{2,0} + a_{2,1}x_1 + \cdots + a_{2,n}x_n &= 0, \\ &\vdots \\ a_{k,0} + a_{k,1}x_1 + \cdots + a_{k,n}x_n &= 0 \end{aligned}$$

of linear equations. A linear combination of elements $c_1e_1 + \cdots + c_ne_n$ is a *solution* to this system if $a_{i,0} + a_{i,1}c_1 + \cdots + a_{i,n}c_n = 0$ for all $i = 1, \ldots, k$. The *solution space* is the collection of elements of $\mathbf{R}^n$ that satisfy all the equations in a system, itself a vector space.

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1. **Warm up 1:** Find at least one solution to each of the following systems of equations, in the appropriate vector spaces. If no solutions exist, say so.

   (a) $5 + 4x_1 = 0$ in $\mathbf{R}^1$

   (b) $5 + 4x_1 = 0$ in $\mathbf{R}^2$

   (c) $\begin{aligned} 2x_1 &= 0 \\ 3x_1 &= 0 \end{aligned}$ in $\mathbf{R}^1$

   (d) $\begin{aligned} 1 + 2x_1 &= 0 \\ 1 + 3x_1 &= 0 \end{aligned}$ in $\mathbf{R}^1$

   (e) $\begin{aligned} 1 + 2x_1 &= 0 \\ 1 + 3x_1 &= 0 \end{aligned}$ in $\mathbf{R}^2$

   (f) $\begin{aligned} 2 + 2x_1 &= 0 \\ 1 - x_2 &= 0 \\ -2 + \pi x_3 &= 0 \end{aligned}$ in $\mathbf{R}^4$

2. **Warm up 2:** Add a new equation to each of the systems above that satisfies the equations already in the system.

A linear system is made up of *non-degenerate* equations, that is, equations that have solutions by itself in the given vector space. For example, $x_1 - 2 - x_1 = 0$ is a degenrate equation, because it simplifies to $1 = 0$. Usually we try to eliminate the *dependent* equations in a linear system, that is, those that are linear combinations of the other equations. Every *independent* equations reduces the solution space by 1 dimension.

To begin with, the vector space $\mathbf{R}^n$ with an empty linear system has a dimension n solution space, as all variables x_i are *independent*, or *free*. Every independent equation in the system makes one of the independent variables *dependent*, though you have a choice as to which becomes dependent.

3. For each of the linear systems below, indicate
   - one solution (if any exist),
   - the independent variables,
   - an independent collection of equations, and
   - the dimension of the solution space.

   (a) $5/2 - 3x_1 = 0$ in $\mathbf{R}^1$

   (b) $5/2 - 3x_1 = 0$ in $\mathbf{R}^3$

   (c)
   $$\begin{aligned} 6 + 2x_2 - x_1 &= 0 \quad \text{in } \mathbf{R}^2 \\ x_1/2 - x_2 &= 3 \end{aligned}$$

   (d)
   $$\begin{aligned} -8x_2 &= 0 \quad \text{in } \mathbf{R}^5 \\ 2 + x_1 + 9x_2 &= 0 \\ x_2 - 2x_5 + 3x_1 + 2 &= 0 \end{aligned}$$

   (e)
   $$\begin{aligned} 2 - 3x_1 + 5x_2 &= 0 \quad \text{in } \mathbf{R}^2 \\ 1 + x_2 &= 0 \end{aligned}$$

   (f)
   $$\begin{aligned} x_1 + 3x_2 - x_4 &= 0 \quad \text{in } \mathbf{R}^4 \\ x_3 - x_1/2 + 4 &= x_3 - 7 \\ 3x_2 + 22 &= x_4 \end{aligned}$$

4. For each of the linear systems above, find
   - all the solutions in the solution space,
   - the matrix that corresponds to the system, and
   - apply row operations to get an identity matrix in the top left corner.