

22 February 2018

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1. This exercise is an introduction to the *Fourier transform* of a function. For a function  $f$  that satisfies  $f(0) = f(1)$ , define its Fourier transform as the function

$$\hat{f}(t) = \int_0^1 f(x)e^{-2\pi itx} dx,$$

where  $i = \sqrt{-1}$  is the *imaginary number*.

- (a) Show that  $\widehat{f'}(x) = 2\pi it\hat{f}(x)$ . Hint: use integration by parts and that  $f(0) = f(1)$ .

- (b) Let  $g$  be some function, and let  $f$  be described by the *differential equation*

$$\frac{f''(x)}{4\pi^2} + 4f(x) = g(x).$$

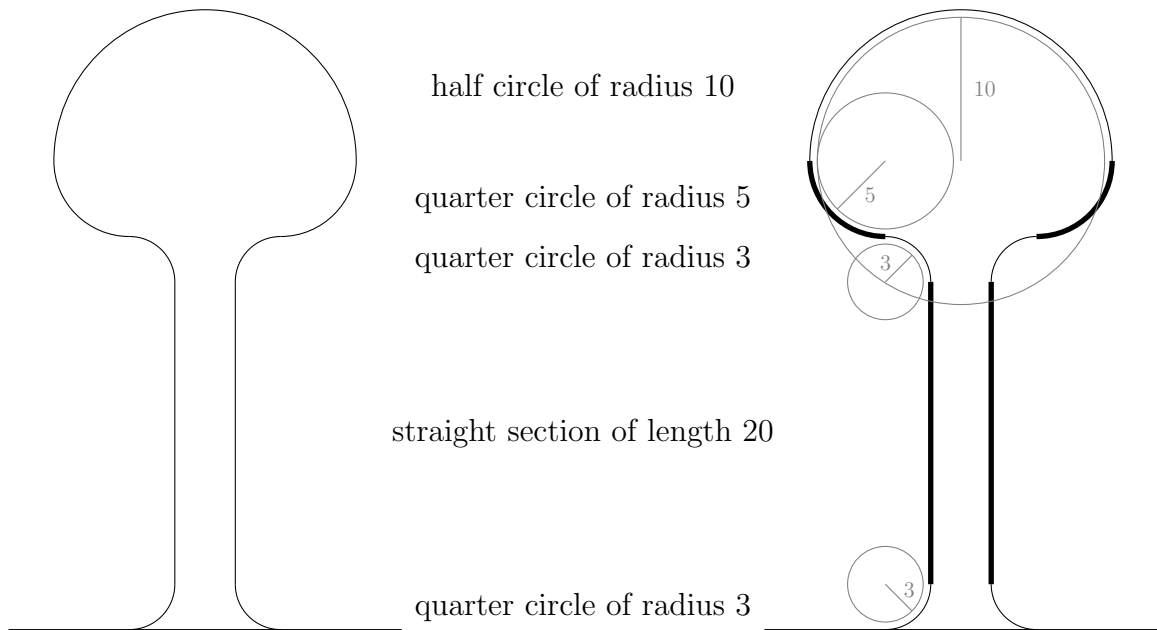
Apply the Fourier transform (take the integral of both sides) to solve for  $\hat{f}$ .

- (c) Let  $f$  be a function defined on  $[0, 1]$  as

$$f(x) = \begin{cases} x & 0 \leq x \leq 1/2, \\ -x + 1 & 1/2 \leq x \leq 1. \end{cases}$$

Calculate the Fourier transform  $\hat{f}$  and find  $\hat{f}(1)$ .

2. Consider a cutaway of a symmetric water tower below, with units in feet.



Given the geometric shapes of the walls, **write down (do not evaluate!)** the integrals and expressions that give the total volume of water that could be contained within the 3-dimensional structure.