ESP Math 182 **Spring 2017**

Mock final - Solutions

27 April 2017

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1. *Integral methods:* Evaluate the following integrals. Show all your work.

   (a) $\displaystyle\int \frac{x^2 e^{\sqrt{x^3-3}}}{\sqrt{x^3-3}}\,dx$

   For this we use integration by substitution, letting $u = \sqrt{x^3-3}$, so $du = \frac{3x^2}{\sqrt{x^3-3}}$ by the chain rule. That is,

$$\int \frac{x^2 e^{\sqrt{x^3-3}}}{\sqrt{x^3-3}}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}\left(e^u + c\right) = \frac{e^u}{3} + c$$

   for some constant c (note that the $1/3$ is absorbed into the constant).

   (b) $\displaystyle\int x^2 \sin(2x-5)\,dx$

   This is integration by parts followed by substitution, with $u = 2x-5$, so $du = 2dx$:

$$\begin{aligned}\int x^2\sin(2x-5)\,dx &= x^2\int \sin(2x-5)\,dx - \int 2x\int \sin(2x-5)\,dx\,dx\\ &= \frac{x^2}{2}\int\sin(u)\,du - \int x\int \sin(u)\,du\,dx\\ &= \frac{x^2(-\cos(u))}{2} + \int x\cos(u)\,dx\\ &= \frac{-x^2\cos(2x-5)}{2} + \int x\cos(2x-5)\,dx,\end{aligned}$$

   and the second term is another integration by parts with the same substitution, as

$$\begin{aligned}\int x\cos(2x-5)\,dx &= x\int\cos(2x-5)\,dx - \int\int\cos(2x-5)\,dx\,dx\\ &= \frac{x}{2}\int\cos(u)\,du - \frac{1}{2}\int\int\cos(u)\,du\,dx\\ &= \frac{x\sin(u)}{2} - \frac{1}{2}\int\sin(u)\,dx\\ &= \frac{x\sin(2x-5)}{2} - \frac{1}{2}\int\sin(2x-5)\,dx\\ &= \frac{x\sin(2x-5)}{2} - \frac{1}{4}\int\sin(u)\,du\\ &= \frac{x\sin(2x-5)}{2} + \frac{\cos(u)}{4}\\ &= \frac{x\sin(2x-5)}{2} + \frac{\cos(2x-5)}{4}\end{aligned}$$

$$= \frac{x}{2}\sin(2x-5) + \frac{1}{4}\cos(2x-5) + c.$$

(c) $\displaystyle\int_5^7 \frac{x+1}{9x^2+4}\,dx$

This is a trigonometric integral, which we notice by the denominator not having any real roots. We want to make this look like $1/\sqrt{u^2+a^2}$ for some u and a, and this is done by splitting it up into two terms and simplifying. Then we substitute $u = 9x^2+4$ (so $du = 18xdx$) in the first term, and $v = 3x$ (so $dv = 3dx$) in the second term to get

$$\begin{aligned}\int \frac{x+1}{9x^2+4}\,dx &= \int \frac{x}{9x^2+4}\,dx + \int \frac{1}{9x^2+4}\,dx \\ &= \frac{1}{18}\int \frac{1}{u}\,du + \frac{1}{3}\int \frac{1}{v^2+2^2}\,dv \\ &= \frac{\ln(|u|)}{18} + \frac{\arctan(v/2)/2}{3} \\ &= \frac{\ln(9x^2+4)}{18} + \frac{\arctan(3x/2)}{6}.\end{aligned}$$

Evaluate this from $x=5$ to $x=7$ to get the answer.

(d) $\displaystyle\int_e^3 \frac{x^2+x-20}{x^3-4x^2+4x}\,dx$

Here we have to use partial fractions. Factoring shows that

$$x^x + x - 20 = (x+5)(x-4), \qquad\qquad x^3-4x^2+4x = x(x-2)^2.$$

Hence we have some constants A, B, C such that

$$\frac{x^2+x-20}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2},$$

or

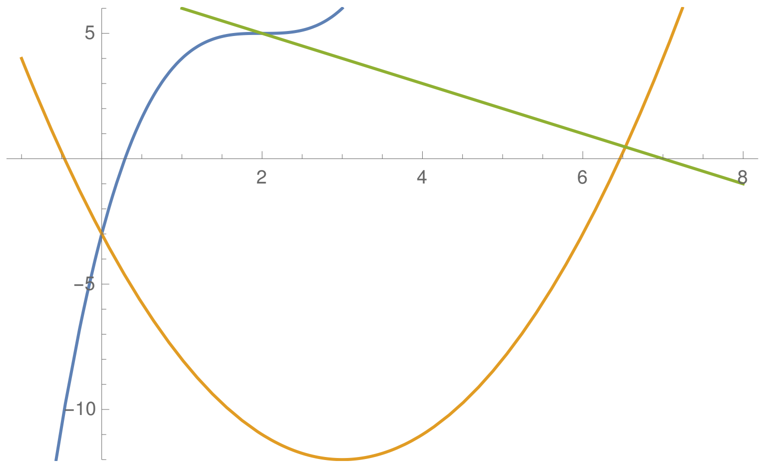
$$x^2+x-20 = A(x-2)^2 + Bx(x-2) + Cx.$$

Evaluating this equation at the points 0 and 2 gives that $-20 = 4A$ and $-14 = 2C$. To get B, we compare coefficients of the x^2 terms on both sides, getting $1 = A+B$. Hence $A=-5$, $B=6$ and $C=-7$. The integral can now be distributed to the terms as follows:

$$\begin{aligned}\int_e^3 \frac{-5}{x}\,dx &= -5\ln(|x|)\Big|_{x=e}^{x=3} = -5\ln(3) - 5, \\ \int_e^3 \frac{6}{x-2}\,dx &= \int_{e-2}^1 \frac{6}{u}\,du = 6\ln(|u|)\Big|_{u=e-2}^{u=1} = -6\ln(e-2), \\ \int_e^3 \frac{-7}{(x-2)^2} & \int_{e-2}^1 \frac{-7}{u^2}\,du = \frac{7}{u}\Big|_{u=e-2}^{u=1} = 7 - \frac{7}{e-2}.\end{aligned}$$

2. *Area between curves:* Find the integral that represents the area above the curve $y = (x-3)^2 - 12$ and below both of the curves $y = (x-2)^3 + 5$ and $y = 7 - x$. Do not evaluate the integral.

The graphs of these three functions on the interval $[-1, 8]$ with range $[-12, 6]$ is below.



The intersection point of the parabola with the cubic function is found to be at

$$\begin{aligned}(x-3)^2 - 12 &= (x-2)^3 + 5\\ x^2 - 6x + 9 - 12 &= x^3 - 6x^2 + 12x - 8 + 5\\ x^3 - 7x^2 + 18x &= 0\\ x(x^2 - 7x + 18) &= 0.\end{aligned}$$

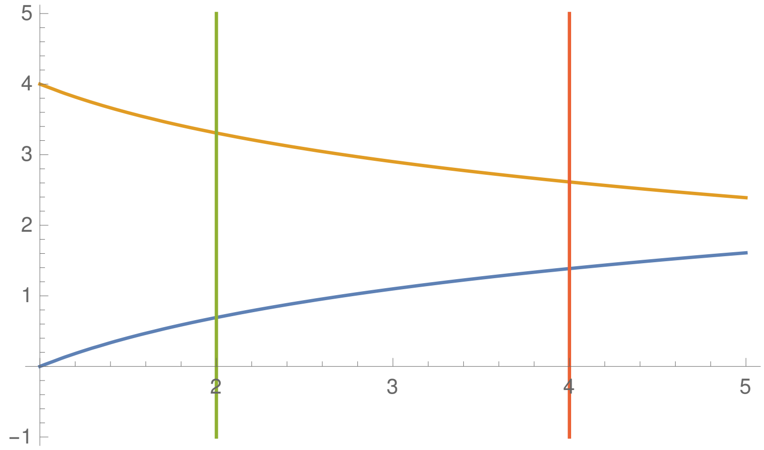
Since $(-7)^2 - 4 \cdot 18 = 49 - 72 = -23 < 0$, we conclude the only solution is $x = 0$, for which $y = -3$. Similarly we find the intersection of the cubic with the line at $(2, 5)$ and the quadratic with the line at $((5+\sqrt{65})/2, (9-\sqrt{65})/2)$. Hence the area of the shape is given by

$$\int_0^2 ((x-2)^3 + 5) - ((x-3)^2 - 12)\ dx + \int_2^{(5+\sqrt{65})/2} (7-x) - ((x-3)^2 - 12)\ dx.$$

3. *Volumes of revolution:* Calculate the following volumes using the disk method.

   (a) The area bounded by $y = \ln(x)$, $y = 4 - \ln(x)$, $x = 2$, and $x = 4$ revolved around the x-axis.

The four curves are given in the diagram below.

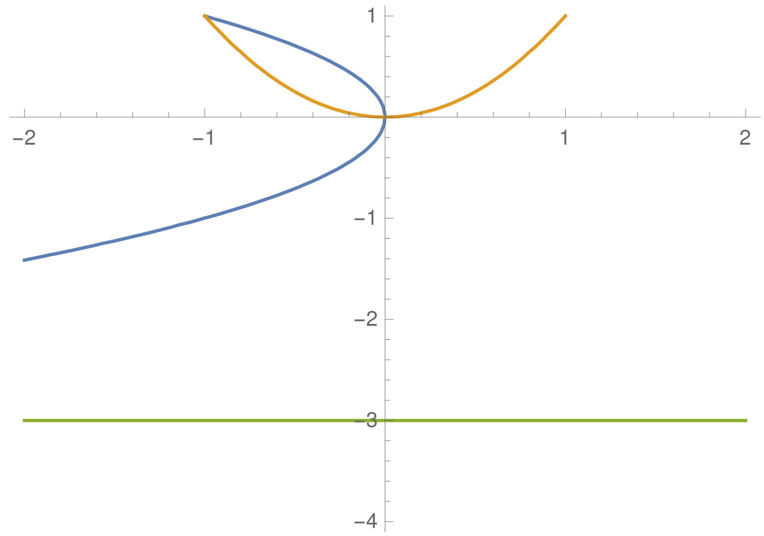


Hence the integral representing the volume is

$$\begin{aligned}\pi\int_2^4 (4-\ln(x))^2-(\ln(x))^2\,dx &= \pi\int_2^4 16-8\ln(x)\,dx\\ &= \pi\left(16x-8(x\ln(x)-x)\right)\Big|_{x=2}^{x=4}\\ &= \pi\left((64-8(4\ln(4)-4))-(32-8(2\ln(2)-2))\right)\\ &= \pi\left(64-32\ln(4)+32-32+16\ln(2)-16\right)\\ &= \pi\left(48-64\ln(2)+16\ln(2)\right)\\ &= 48\pi(1-\ln(2)).\end{aligned}$$

(b) The area in the second quadrant bounded by $x=-y^2$ and $y=x^2$ revolved around the axis $y=-3$.

The two curves and the axis $y=-3$ are given the in the diaram below.

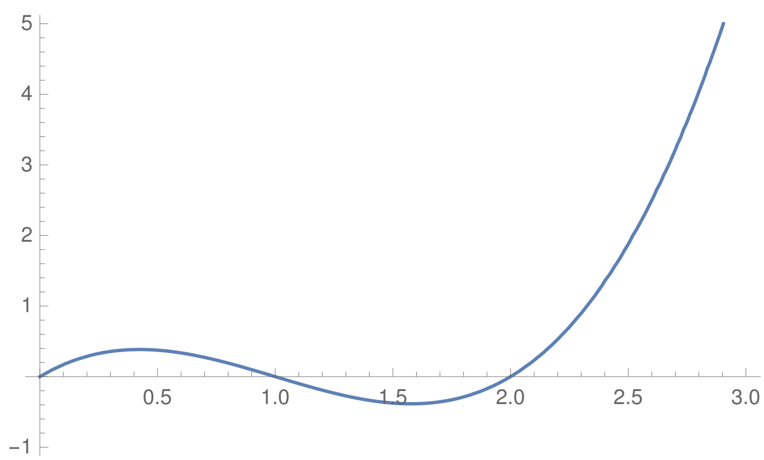


Expressing the curve $x=-y^2$ in terms of x we get $y=\pm\sqrt{-x}$. We chose the positive side $+\sqrt{-x}$, because that is the one above the x-axis. Since we are rotating not around the x-axis, but around a line shifted three units below the x-axis, we have to add 3 to both functions to get the right shape. Hence the integral representing the volume is

$$\begin{aligned}
\pi \int_{-1}^{0} (\sqrt{-x}+3)^2 - (x^2+3)^2 \; dx &= \pi \int_{-1}^{0} -x + 3\sqrt{-x} + 9 - x^4 - 6x^2 - 9 \; dx \\
&= \pi \left(-\int_{-1}^{0} x \; dx + 3 \int_{-1}^{0} \sqrt{-x} \; dx - \int_{-1}^{0} x^4 \; dx - 6 \int_{-1}^{0} x^2 \; dx \right) \\
&= \pi \left(\frac{x^2}{2} + 2(-x)^{3/2} - \frac{x^5}{5} - 2x^3 \right) \Bigg|_{x=-1}^{x=0} \\
&= \pi \left(\frac{(-1)^2}{2} + 2(1)^{3/2} - \frac{(-1)^5}{5} - 2(-1)^3 \right) \\
&= \pi \left(\frac{1}{2} + 2 + \frac{1}{5} + 2 \right) \\
&= \frac{47\pi}{10}.
\end{aligned}$$

(c) The volume of revolution of $y = x(x-1)(x-2)$ revolved around the x-axis between $x = 0$ and $x = 3$.

The curve is given in the diagram below.



Here we simply integrate from 1 to 3 with the height of the function as the radius of the disks. So the volume of the solid is

$$\pi \int_{0}^{3} (x(x-1)(x-2))^2 \; dx = \frac{288\pi}{35}.$$

The calculations are skipped because the integrand is just a polynomial, with no tricks.

4. *Sequences:* For each of the following sequences, determine if it converges or diverges. If it converges find the limit.

(a) $x_n = \dfrac{n}{n+1}$

Observe that

$$\lim_{n\to\infty}\left[\frac{n}{n+1}\right] = \lim_{n\to\infty}\left[\frac{\frac{1}{n}}{\frac{1}{n}}\cdot\frac{n}{n+1}\right] = \lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}}\right] = \frac{1}{1+0} = 1,$$

so the sequence converges, and converges to 1.

(b) $x_n = \dfrac{n\cos(n\pi)}{2n+1}$

Observe that when n is an odd number, $\cos(n\pi) = \cos(\pi) = -1$, so then

$$\lim_{n\to\infty}\left[\frac{n\cos(n\pi)}{2n+1}\right] = \lim_{n\to\infty}\left[\frac{-n}{2n+1}\right] = \lim_{n\to\infty}\left[\frac{-1}{2+\frac{1}{n}}\right] = \frac{-1}{2+0} = -\frac{1}{2},$$

but if n is odd, then $\cos(n\pi) = \cos(0) = 1$, so then

$$\lim_{n\to\infty}\left[\frac{n\cos(n\pi)}{2n+1}\right] = \lim_{n\to\infty}\left[\frac{n}{2n+1}\right] = \lim_{n\to\infty}\left[\frac{1}{2+\frac{1}{n}}\right] = \frac{1}{2+0} = \frac{1}{2},$$

so the limits are not the same. That is, the sequence alternates between $1/2$ and $-1/2$ forever. Hence the sequence does not converge.

(c) $x_n = \dfrac{\sin(n)}{n}$

This is an application of the squeeze theorem. Recall that $-1 \leqslant \sin(x) \leqslant 1$ for any argument x, so then

$$\begin{aligned}
-1 \leqslant \sin(n) &\leqslant 1\\
-\frac{1}{n} \leqslant \frac{\sin(n)}{n} &\leqslant \frac{1}{n}\\
\lim_{n\to\infty}\left[-\frac{1}{n}\right] \leqslant \lim_{n\to\infty}\left[\frac{\sin(n)}{n}\right] &\leqslant \lim_{n\to\infty}\left[\frac{1}{n}\right]\\
-0 \leqslant \lim_{n\to\infty}\left[\frac{\sin(n)}{n}\right] &\leqslant 0.
\end{aligned}$$

Hence the sequence $\sin(n)/n$ converges to 0.

5. *Series:* Find the intervals of convergence of the following series. Indicate which tests you have used.

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{(x-2)^n}{(n\ln(n))^2}$

Note that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(x-2)^{n+1}}{((n+1)\ln(n+1))^2}}{\frac{(x-2)^n}{(n\ln(n))^2}} = \frac{(x-2)(n\ln(n))^2}{((n+1)\ln(n))^2}.$$

We only take the factors inside the square as $n \to \infty$, and apply l'Hopital's rule to get that

$$\lim_{n\to\infty}\left[\frac{n\ln(n)}{(n+1)\ln(n)}\right] = \lim_{n\to\infty}\left[\frac{\ln(n)+\frac{n}{n}}{\ln(n)+\frac{n}{n}+\frac{1}{n}}\right] = \lim_{n\to\infty}\left[\frac{1+\frac{1}{\ln(n)}}{1+\frac{1}{\ln(n)}+\frac{1}{n\ln(n)}}\right] = 1.$$

The square of the limit also goes to 1, so the series certainly converges for $|x-2|<1$, or $1<x<3$. At the endpoints, we have that

$$\frac{(3-2)^n}{(n\ln(n))^2} = \frac{1}{(n\ln(n))^2} \leqslant \frac{1}{n^2},$$

which converges by the p-series test. The other endpoint converges by the alternating series test, so the interval of convergence for this series is $x \in [1,3]$.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{15^n n}$

Note that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(x-3)^{n+1}}{15^{n+1}(n+1)}}{\frac{(x-3)^n}{15^n n}} = \frac{(x-3)n}{15(n+1)},$$

and taking the limit of this as $n \to \infty$, we get

$$\lim_{n\to\infty}\left[\frac{(x-3)n}{15(n+1)}\right] = \frac{x-3}{15}\lim_{n\to\infty}\left[\frac{n}{n+1}\right] = \frac{x-3}{15}\lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}}\right] = \frac{x-3}{15},$$

so by the ratio test, we have that the series definitely converges for $|\frac{x-3}{15}|<1$, or $-12<x<18$. For the endpoints, we have that

$$\frac{(18-3)^n}{15^n n} = \frac{1}{n}, \qquad \frac{(-12-3)^n}{15^n n} = \frac{(-1)^n}{n},$$

both of which are divergent series. Hence the interval of convergence is $x \in (-12,18)$ for this series.

6. *Parametric equations:*

(a) Describe the linear system

$$\begin{aligned} 4x+5y-2z &= 7, \\ x-y+10z &= 1 \end{aligned}$$

as a parametric equation in the variable t.

We choose z to be our free variable (but any other would work). Solve the second equation for x (as that is easier) and replace it in the first to get

$$4(1+y-10z)+5y-2z = 7 \quad\Longrightarrow\quad y = \frac{42z+3}{9}.$$

Set $t = z$ and replace this in the second equation to get

$$x = 1 + \frac{42t + 3}{9} + 10t = \frac{132t + 12}{9}.$$

Hence the parametric equation describing this linear system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 132/9 \\ 42/9 \\ 1 \end{pmatrix} t + \begin{pmatrix} 4/3 \\ 1/3 \\ 0 \end{pmatrix}.$$

(b) For the parametric curve $(x, y) = (5t - 2, 8 - 3t)$, find $\frac{dy}{dx}$ and the values of t for which the graph is in the first quadrant.

Recall that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3}{5}.$$

For the first quadrant, we must have both the x-values and y-values be positive, so

$$\begin{aligned} 5 - 2t &\geqslant 0 \\ 5t &\geqslant 2 \\ t &\geqslant 2/5, \end{aligned} \qquad\qquad \begin{aligned} 8 - 3t &\geqslant 0 \\ 8 &\geqslant 3t \\ 8/3 &\geqslant t. \end{aligned}$$

In other words, we must have $2/5 \leqslant t \leqslant 8/3$ for the graph to be in the first quadrant.

7. *Matrices:* Find the determinant, eigenvalues, and eigenvectors of the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right]$.

The determinant is $1 \cdot 2 - 1 \cdot 2 = 0$. Recall the eigenvalues of a matrix A are the roots of the polynomial $\det(A - \lambda x) = 0$. That is,

$$0 = \det \begin{bmatrix} 1 - \lambda & 1 \\ 2 & 2 - \lambda \end{bmatrix} = (1 - \lambda)(2 - \lambda) - 2 = \lambda^2 - 3\lambda + 2 - 2 = \lambda(\lambda - 3).$$

It is immediate that one of the eigenvalues is 0 and the other is 3. For the 0 eigenvalue, we must have

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or $a + b = 0$. One easy choice is $a = 1$ and $b = -1$. For the eigenvalue 3 we similarly get the equation $a + b = 3a$, or $b = 2a$, for which an easy choice is $a = 1$ and $b = 2$. Hence we get eigenvalue and eigenvector pairs

$$0, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad\qquad 3, \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$