ESP Math 182 | **Spring 2017**

Worksheet 25

11 April 2017

Recall that an $m \times n$ *matrix* A is a collection of mn elements, represented by A_{ij} for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

- The *sum* of two $m \times n$ matrices A, B is an $m \times n$ matrix $C_{ij} = A_{ij} + B_{ij}$
- The *product* of an $m \times n$ matrix A and an $n \times r$ matrix B is a $m \times r$ matrix $C_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}$
- The *transpose* of an $m \times n$ matrix A is an $n \times m$ matrix $(A^T)_{ij} = A_{ji}$
- The *inverse* of an $n \times n$ matrix A is an $n \times n$ matrix A^{-1} such that $AA^{-1} = I_n$
- To a matrix A we may apply *elementary row operations* to its rows R_k:
  - $R_k \to cR_k$, for $c \neq 0$
  - $R_k \to R_\ell$ and $R_\ell \to R_k$
  - $cR_k + R_\ell \to R_\ell$
- The *reduced echelon form* of an $m \times n$ matrix A is the $m \times n$ matrix $R = [I_m|B]$, for B an $m \times (n-m)$ matrix, and R obtained from A by elementary row operations and column swapping. We assume $n > m$.

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1. **Warm up 1:** Find coefficient vectors $\vec{x}$ that make the equalities true.

   (a) $$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

   (b) $$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

2. **Warm up 2:** Turn the linear systems, in the given vector space, into augmented matrices.

   (a) $$\begin{aligned} x_1 - 3x_2 &= 5 \quad \text{in } \mathbf{R}^3 \\ 9 - x_2 - x_3 &= x_1 + 2 \end{aligned}$$

   (b) $$\begin{aligned} 2 + 2x_1 &= x_4 \quad \text{in } \mathbf{R}^4 \\ 1 - x_2 + 5x_1 &= 7 \\ -2 + \pi x_4 &= 0 \end{aligned}$$

3. By elementary row operations, bring the following matrix to a reduced echelon matrix (that is, make it look like the matrix on the right). Show the row operations that you carry out.

$$\begin{bmatrix} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix}$$

4. This question will explore properties of the space of all matrices.

   (a) Find two 2×2 matrices A, B such that $AB \neq BA$. This means matrices are not *commutative.*

   (b) Find two 2×2 matrices C, D such that $CD = 0$, but $C \neq 0$ and $D \neq 0$. This means matrices are not an *integral domain.*

   (c) Show that for all 2×2 matrices A, B, C, we have $(AB)C = A(BC)$. This means that matrices are *associative.*

The *determinant* of a square matrix is a number that is 0 if the matrix does not have an inverse, and nonzero otherwise. The general formula is complicated, but for small matrices we have

$$\det \begin{bmatrix} a \end{bmatrix} = a, \qquad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc, \qquad \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

5. The *rotation matrix* $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ rotates any 2-vectors by an angle of θ.

   (a) What is the determinant of the rotation matrix?

   (b) Calculate R^2.

   (c) Find a matrix S such that $S^2 = R$ (the square root of R). *Hint: think geometrically.*