

9 March 2017

1. **Warm up:** Let $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

(a) Find $f'(0)$, $f''(0)$, and $f'''(0)$.

(b) For what values of x is $f(x) = 0$?

(c) Make a guess as to what is the Taylor series of f at 0.

2. Consider the series $f(x) = \sum_{n=0}^{\infty} \frac{(2x+3)^n}{n!}$.

(a) Find the derivative $f'(x)$ of the series and rewrite it in terms of $f(x)$.

(b) Using part (a), give the n th derivative of $f(x)$. Do not simply keep taking derivatives of the series.

(c) What common function is $f(x)$ equal to?

3. Let $f(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}$.

(a) Show that $f(x)$ has infinite radius of convergence.

(b) Show that $f'(x) = xf(x)$.

4. Recall that $f^{(k)}(x)$ means the k th derivative of f .

(a) Complete the following expressions:

$$\sin^{(k)}(x) = \begin{cases} \text{_____} & \text{if } k = 4n, \\ \text{_____} & \text{if } k = 4n + 1, \\ \text{_____} & \text{if } k = 4n + 2, \\ \text{_____} & \text{if } k = 4n + 3, \end{cases} \quad \cos^{(k)}(x) = \begin{cases} \text{_____} & \text{if } k = 4n, \\ \text{_____} & \text{if } k = 4n + 1, \\ \text{_____} & \text{if } k = 4n + 2, \\ \text{_____} & \text{if } k = 4n + 3, \end{cases}$$

where n is any integer.

(b) Use part (a) to give the Taylor series for $\sin(x)$ and $\cos(x)$ at $x = 0$.

(c) Use part (b) to decide if the following series converges, and if yes, to what:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1} (2n+1)!}$$

5. Find power series representations for the following functions.

(a) $\frac{1}{1-x}$

(c) $\frac{x}{1-x}$

(b) $\frac{1}{1-x^2}$

(d) $\frac{x^2}{1-x^3}$