Calculus I # Integrals and word problems (solutions) **Fall 2014**

Discussion session 21 - 4 November 2014

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1. Take the derivative and indefinite integral, with respect to x, of the following functions:

   (a) x
   Solution:
   $$\frac{d}{dx}(x) = 1 \qquad \int x dx = \frac{x^2}{2} + C$$

   (b) e
   Solution:
   $$\frac{d}{dx}(e) = 0 \qquad \int e dx = ex + C$$

   (c) x^e
   Solution:
   $$\frac{d}{dx}(x^e) = ex^{e-1} \qquad \int x^e dx = \frac{x^{e+1}}{e+1} + C$$

   (d) e^x
   Solution:
   $$\frac{d}{dx}(e^x) = e^x \qquad \int e^x dx = e^x + C$$

   (e) e^e
   Solution:
   $$\frac{d}{dx}(e^e) = 0 \qquad \int e^e dx = e^e x + C$$

   (f) e^{x^e}
   Solution:
   $$\frac{d}{dx}(e^{x^e}) = e^{x^e} ex^{e-1} = e^{x^e+1}x^{e-1}$$
   The integral is too difficult for you to calculate. ∎

2. Here are some useful identities:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad\qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Use these identities to help simplify the following expressions (that is, find an answer only in terms of n).

(a) $\sum_{k=1}^{n}(2k+3k^2)$

Solution:

$$\begin{aligned}\sum_{k=1}^{n}(2k+3k^2) &= \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 3k^2 \\ &= 2\sum_{k=1}^{n} k + 3\sum_{k=1}^{n} k^2 \\ &= 2\frac{n(n+1)}{2} + 3\frac{n(n+1)(2n+1)}{6} \\ &= n(n+1) + \frac{n(n+1)(2n+1)}{2}.\end{aligned}$$

(b) $\sum_{j=0}^{n-3} \frac{j^2+2j+1}{2j+2}$

Solution:

$$\begin{aligned}\sum_{j=0}^{n-3} \frac{j^2+2j+1}{2j+2} &= \sum_{j=0}^{n-3} \frac{(j+1)(j+1)}{2(j+1)} \\ &= \sum_{j=0}^{n-3} \frac{j+1}{2} \\ &= \sum_{j=0}^{n-3} \left(\frac{j}{2}+\frac{1}{2}\right) \\ &= \sum_{j=0}^{n-3} \frac{j}{2} + \sum_{j=0}^{n-3} \frac{1}{2} \\ &= \frac{1}{2}\sum_{j=0}^{n-3} j + \frac{1}{2}\sum_{j=0}^{n-3} 1 \\ &= \frac{1}{2}\frac{(n-3)(n-3+1)}{2} + \frac{1}{2}(n-3) \\ &= \frac{(n-3)(n-2)}{4} + \frac{n-3}{2}.\end{aligned}$$

(c) $\displaystyle\sum_{k=2}^{n+2} \big((k-2)^3 + k^2\big) + \sum_{k=3}^{n+3} \big((k-3)^3 - 2k\big) - \sum_{k=0}^{n} 2k^3$

Solution:

$$\begin{aligned}
(\text{sum}) &= \sum_{j=0}^{n} \big(j^3 + (j+2)^2\big) + \sum_{k=3}^{n+3} \big((k-3)^3 - 2k\big) - \sum_{k=0}^{n} 2k^3 && (\text{set } j = k-2)\\
&= \sum_{j=0}^{n} \big(j^3 + (j+2)^2\big) + \sum_{\ell=0}^{n} \big(\ell^3 - 2(\ell+3)\big) - \sum_{k=0}^{n} 2k^3 && (\text{set } \ell = k-3)\\
&= \sum_{k=0}^{n} \big(k^3 + (k+2)^2\big) + \sum_{k=0}^{n} \big(k^3 - 2(k+3)\big) - \sum_{k=0}^{n} 2k^3 && (\text{rename } j \text{ and } \ell \text{ as } k)\\
&= \sum_{k=0}^{n} \big(k^3 + (k+2)^2 + k^3 - 2(k+3) - 2k^3\big)\\
&= \sum_{k=0}^{n} \big(k^2 + 4k + 4 - 2k - 6\big)\\
&= \sum_{k=0}^{n} \big(k^2 + 2k - 2\big)\\
&= \sum_{k=0}^{n} k^2 + 2\sum_{k=0}^{n} k - 2\sum_{k=0}^{n} 1\\
&= \frac{n(n+1)(2n+1)}{6} + n(n+1) - 2n.
\end{aligned}$$

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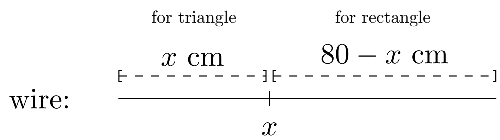
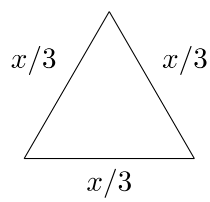
3. An 80cm piece of wire is cut into two pieces. One piece is bent into an equilateral triangle. The other piece is bent into a rectangle with one side four times the length of the other side. Determine where, if anywhere, the wire should be cut to maximize the area enclosed by the two figures.

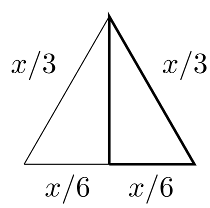
Solution: Let x be the position on the wire, with $x = 0$ the left end, where it will be cut. The first part of the wire will go to the triangle, and the second part will go to the rectangle. So the wire may be partitioned into two pieces as below, with lengths as indicated.



We now will find the area function $A(x)$, which will only depend on one variable. The area of a triangle is $\frac{1}{2}$(base)(height). Since the triangle is equilateral, we know that all sides have equal length. Since the perimeter is x by the condition above, each side of the triangle has length $x/3$. So the situation looks as in the diagram below.



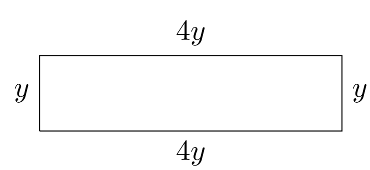
The triangle has base $x/3$, but the height is unknown. Drawing in the height, we have two right triangles, one of which is emphasized, as in the diagram below.



Apply the Pythagorean theorem to the emphasized triangle to get that

$$(\text{height})^2 = \left(\frac{x}{3}\right)^2 - \left(\frac{x}{6}\right)^2 = \frac{x^2}{9} - \frac{x^2}{36} = \frac{4x^2 - x^2}{36} = \frac{3x^2}{36} = \frac{x^2}{12}.$$

Therefore the triangle has height $= \frac{x}{2\sqrt{3}}$, and area $\frac{1}{2} \cdot \frac{x}{3} \cdot \frac{x}{2\sqrt{3}} = \frac{x^2}{12\sqrt{3}}$. We now find the area of the rectangle. Let y be the length of the shorter side of the rectangle. Recall that the length of the longer side is four times the length of the shorter side, so we have a situation as in the diagram below.



The perimeter of the rectangle is $y + y + 4y + 4y = 10y$, and the area is $4y \cdot y = 4y^2$. From the first diagram above, we have $80 - x$ cm for the perimeter of the rectangle, so we have $80 - x = 10y$, or $y = \frac{80-x}{10}$. Therefore, in terms of x, the area of the rectangle is $4\left(\frac{80-x}{10}\right)^2$. Hence the total area of the two shapes is

$$\begin{aligned} A(x) &= \frac{x^2}{12\sqrt{3}} + 4\left(\frac{80-x}{10}\right)^2 \\ &= \frac{x^2}{12\sqrt{3}} + \frac{4(6400 - 160x + x^2)}{100} \\ &= x^2\left(\frac{1}{12\sqrt{3}} + \frac{4}{100}\right) + x\left(\frac{-4 \cdot 160}{100}\right) + \frac{4 \cdot 6400}{100} \\ &= x^2\left(\frac{1}{12\sqrt{3}} + \frac{1}{25}\right) + x\left(\frac{-32}{5}\right) + 256. \end{aligned}$$

We would like to maximize this function to find the maximum area. To maximize, we must find the critical points, which means the derivative has to be zero. The derivative is

$$A'(x) = 2x\left(\frac{1}{12\sqrt{3}} + \frac{1}{25}\right) + \left(\frac{-32}{5}\right),$$

and setting it equal to zero, we find that

$$0 = 2x\left(\frac{1}{12\sqrt{3}} + \frac{1}{25}\right) + \left(\frac{-32}{5}\right) \quad \Longrightarrow \quad x = \frac{\frac{32}{5}}{\frac{2}{12\sqrt{3}} + \frac{2}{25}} = \frac{32 \cdot 60\sqrt{3}}{50 + 24\sqrt{3}} \approx 36.32.$$

However, this is not guaranteed to be a maximum, only a critical point. We construct a monotonicity table to find if it is the maximum of minimum.

		36.32	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

Therefore $x = 36.32$ is a minimum, which means that we would get the least area if we cut it at that spot. Since there are no other critical points, we must check the endpoints of the interval, which is $[0, 80]$ by the conditions given. Cutting the wire at $x = 0$ means giving the whole wire to the rectangle, and cutting at $x = 80$ means giving the whole wire to the triangle. So:

area of triangle with 80cm perimeter: $\dfrac{80^2}{12\sqrt{3}} \approx 307.92,$

area of rectangle with 80cm perimeter: $4\left(\dfrac{80}{10}\right)^2 = 256.$

Therefore the largest area results in giving the triangle all the wire, meaning the wire should not be cut at all (equivalently, the cut should be made at $x = 80$ cm). ■