SURPRISE FINAL EXAM

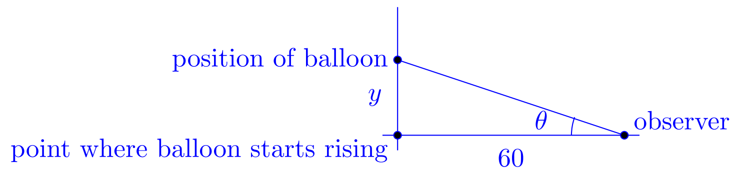
Dec. 1, 2015

Solutions

To express your gratitude to your TA for providing these solutions, go and fill out the TA evaluation online!

1. A balloon rises at a rate of 1 ft/sec from a point on the ground 60 ft away from an observer. Find the rate of change of the angle of elevation to the balloon from the observer when the balloon is 80 ft above the ground.

The situation is as in the diagram below.



Since tangent=opposite/adjacent, take the tangent of θ to get

$$\tan(\theta) = \frac{y}{60} \qquad \Longrightarrow \qquad \theta = \tan^{-1}\left(\frac{y}{60}\right).$$

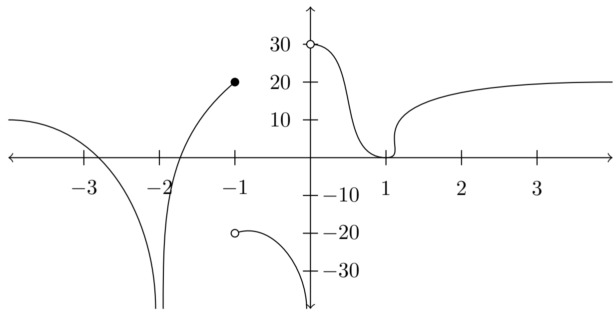
We already know $\frac{dy}{dt} = 1$ by assumption, so taking the derivative of both sides of the above equation with respect to t, we get

$$\begin{aligned}
\frac{d}{dt}(\theta) &= \frac{d}{dt}\left(\tan^{-1}\left(\frac{y}{60}\right)\right) \\
\frac{d\theta}{dt} &= \frac{1}{1+\left(\frac{y}{60}\right)^2}\frac{d}{dt}\left(\frac{y}{60}\right) \\
\frac{d\theta}{dt} &= \frac{1}{60\left(1+\frac{y^2}{60^2}\right)} \cdot \frac{dy}{dt} \\
\frac{d\theta}{dt} &= \frac{1}{60+\frac{y^2}{60}} \cdot 1.
\end{aligned}$$

Repace the y with 80 (because that is what the question is asking) to get

$$\left.\frac{d\theta}{dt}\right|_{y=80} = \frac{1}{60+\frac{80^2}{60}} = \frac{60}{3600+6400} = \frac{60}{10000} = \frac{3}{500} \text{ radians/second.}$$

2. Use the graph shown here to answer the questions below.



(a) $\lim_{x\to\infty} f(x) = 20$

(b) $\lim_{x\to 1^-} f(x) = 0$

(c) $\lim_{x\to -1} f(x)$ DNE (does not exist)

(d) $\lim_{x\to -2^-} f(x)$ DNE or $-\infty$

(e) $\lim_{x\to 0^+} f(x) = 30$

(f) $f(-1) = 20$

(g) $f(0)$ DNE

3. Evaluate the following limits.

(a) $\lim_{x\to 0^+} \frac{5-\sqrt{25+x}}{x\sqrt{25+x}}$

$$\lim_{x\to 0^+} \frac{5-\sqrt{25+x}}{x\sqrt{25+x}} = \lim_{x\to 0^+} \frac{5-\sqrt{25+x}}{x\sqrt{25+x}} \cdot \frac{5+\sqrt{25+x}}{5+\sqrt{25+x}} = \lim_{x\to 0^+} \frac{25-25-x}{x\sqrt{25+x}(5+\sqrt{25+x})}$$
$$= \lim_{x\to 0^+} \frac{-x}{x\sqrt{25+x}(5+\sqrt{25+x})} = \lim_{x\to 0^+} \frac{-1}{\sqrt{25+x}(5+\sqrt{25+x})}$$
$$= \frac{-1}{\sqrt{25}(5+\sqrt{25})} = \frac{-1}{50}$$

(b) $\lim_{x\to 3} \frac{x^3-9x}{3-x}$

$$\lim_{x\to 3} \frac{x^3-9x}{3-x} = \lim_{x\to 3} \frac{x(x^2-9)}{3-x} = \lim_{x\to 3} \frac{x(x+3)(x-3)}{-(x-3)} = \lim_{x\to 3} -x(x+3) = -3(3+3) = -18$$

(c) $\lim_{x\to 1} \frac{1+\cos(\pi x)}{2-2x+2\ln(x)}$

$$\lim_{x\to 1} \frac{1+\cos(\pi x)}{2-2x+2\ln(x)} = \frac{0}{0}\ L'H$$
$$= \lim_{x\to 1} \frac{-\pi\sin(\pi x)}{-2+\frac{2}{x}} = \lim_{x\to 1} \frac{-\pi\sin(\pi x)}{\frac{-2x+2}{x}} = \lim_{x\to 1} \frac{-\pi x\sin(\pi x)}{-2x+2} = \frac{0}{0}\ L'H$$
$$= \lim_{x\to 1} \frac{-\pi\sin(\pi x) - \pi^2 x\cos(\pi x)}{-2} = \frac{-\pi^2}{2}$$

(d) $\lim_{x\to 3} \frac{-3}{3+x}$

$$\lim_{x\to 3} \frac{-3}{3+x} = \frac{-3}{3+3} = \frac{-3}{6} = \frac{-1}{2}$$

4. Find $\frac{dy}{dx}$ for the following functions. Do **NOT** simplify your answer.

(a) $y = \frac{7^{7x} + \ln(3x^2+2)}{3 - \csc(5x) + \frac{1}{\sqrt{x}}}$

$$\frac{\left(3 - \csc(5x) + \frac{1}{\sqrt{x}}\right)\left(7^{7x}\ln(7)7 + \frac{6x}{3x^2+2}\right) - \left(7^{7x} + \ln(3x^2+2)\right)\left(\csc(5x)\cot(5x)5 - \frac{1}{2}x^{-3/2}\right)}{\left(3 - \csc(5x) + \frac{1}{\sqrt{x}}\right)^2}$$

(b) $2x^3y^2 + 4x^2 = 3y^3 - 5x\ln(y)$

$$6x^2y^2 + 2x^3 2y\frac{dy}{dx} + 8x = 9y^2\frac{dy}{dx} - 5\ln(y) - 5x\frac{1}{y}\frac{dy}{dx}$$

$$4x^3y\frac{dy}{dx} - 9y^2\frac{dy}{dx} + 5\frac{x}{y}\frac{dy}{dx} = -6x^2y^2 - 8x - 5\ln(y)$$

$$\frac{dy}{dx}\left(4x^3y - 9y^2 + 5\frac{x}{y}\right) = -6x^2y^2 - 8x - 5\ln(y)$$

$$\frac{dy}{dx} = \frac{-6x^2y^2 - 8x - 5\ln(y)}{4x^3y - 9y^2 + 5\frac{x}{y}}$$

5. Using the limit definition of the derivative, find the derivative of $f(x) = 3x^2 + 4x - 5$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{3(x+h)^2 + 4(x+h) - 5 - 3x^2 - 4x + 5}{h} \\ &= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 4x + 4h - 5 - 3x^2 - 4x + 5}{h} \\ &= \lim_{h\to 0} \frac{6xh + 3h^2 + 4h}{h} \\ &= \lim_{h\to 0} 6x + 3h + 4 \\ &= 6x + 4 \end{aligned}$$

6. Answer the following questions using the function $f(x) = \frac{1}{4}x^4 + 4x^3 + 50$ on the interval $[-15, 15]$.

(a) On what intervals is the function **increasing**?

To find where f is increasing (and decreasing), we need to find the critical points of f, which are the zeros of the derivative. The derivative of f is $f'(x) = x^3 + 12x^2$, and setting it equal to zero we get

$$x^3 + 12x^2 = 0 \quad \Longrightarrow \quad x^2(x+12) = 0 \quad \Longrightarrow \quad x = 0 \text{ or } x = -12.$$

Now make a table to see what f looks like between these points and the endpoints (by taking test values in the intervals).

interval	$(-15, -12)$	$(-12, 0)$	$(0, 15)$
sign of f'	$-$	$+$	$+$
shape of f	$\searrow$	$\nearrow$	$\nearrow$

Therefore f is increasing on the intervals $(-12, 0)$ and $(0, 15)$.

(b) On what intervals is the function **concave down**?

To find the intervals of concavity, we need to find the critical points of the first derivative, which are zeros of the second derivative. The second derivative of f is $f''(x) = 3x^2 + 24x$, and setting it equal to zero we get

$$3x^2 + 24x = 0 \quad \Longrightarrow \quad 3x(x+8) = 0 \quad \Longrightarrow \quad x = 0 \text{ or } x = -8.$$

Again we make a table with intervals between the found points and the endpoints, and take test value, just as in part (a).

interval	$(-15, -8)$	$(-8, 0)$	$(0, 15)$
sign of f''	$+$	$-$	$+$
shape of f	$\smile$	$\frown$	$\smile$

Therefore f is concave down on the interval $(-8, 0)$.

(c) Give the exact values of the **local extrema** and specify where they occur. Be sure to indicate whether the extrema are minima or maxima or neither.

We have already done the work in part (a), since local extrema are where the derivative is zero and the second derivative is not zero (a point of inflection is not a local extremum). We find $f(-12) = -1678$, so f has one local extremum, a local minimum of -1678 at $x = -12$.

(d) Give the exact values of the **inflection points** and specify where they occur.

We have already done the work in part (b), since local extrema are where the second derivative is zero. We find $f'(-8) = 256$ and $f(0) = 0$, so f has two inflection points of 256 and 0 at $x = -8$ and $x = 0$, respectively.

(e) Give the exact values of the **absolute extrema** and specify where they occur. Be sure to indicate whether the extrema are minima or maxima or neither.

Here we compare the values of f at the local extreme and at the endpoints. We find the following:

x	-15	-12	15
$f(x)$	-793.75	-1678	26206.3

Therefore on the interval $[-15, 15]$, f has a global maximum of 26206.3 at $x = 15$ and a global minimum of -1678 at $x = 12$.

7. Compute the following definite/indefinite integrals.

(a) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}} + 6x^2 + 5\ dx$

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}} + 6x^2 + 5\ dx &= \int_1^4 x^{-1/2} + 6x^2 + 5\ dx = 2x^{1/2} + 2x^3 + 5x\Big|_1^4 \\ &= 2\sqrt{4} + 2(4)^3 + 5(4) - (2\sqrt{1} + 2(1)^3 + 5(1)) \\ &= 4 + 128 + 20 - 2 - 2 - 5 = 143\end{aligned}$$

(b) $\displaystyle\int \frac{x}{\sqrt{x-10}}\ dx$

Let $u = x - 10$. Then $x = u + 10$ and $du = dx$ so,

$$\begin{aligned}\int \frac{x}{\sqrt{x-10}}\ dx = \int \frac{u+10}{\sqrt{u}}\ du = \int u^{1/2} + 10u^{-1/2}\ du &= \frac{2}{3}u^{3/2} + 20u^{1/2} + C \\ &= \frac{2}{3}(x-10)^{3/2} + 20(x-10)^{1/2} + C\end{aligned}$$

(c) $\displaystyle\int \frac{1}{x^2 + 2x + 10}\ dx$ (complete the square)

We first complete the square:

$$x^2 + 2x + 10 \ = \ x^2 + 2x + 1 + 10 - 1 \ = \ (x+1)^2 + 9.$$

So the integral becomes

$$\int \frac{1}{x^2 + 2x + 10}\ dx = \int \frac{1}{(x+1)^2 + 9}\ dx = \int \frac{1}{9\left(\frac{(x+1)^2}{9} + 1\right)} = \frac{1}{9}\int \frac{1}{\left(\frac{x+1}{3}\right)^2 + 1}\ dx.$$

Let $u = \frac{x+1}{3}$, then $du = \frac{1}{3}dx$ so we make the substitution

$$\frac{1}{9}\int \frac{1}{\left(\frac{x+1}{3}\right)^2 + 1} = \frac{3}{9}\int \frac{1}{u^2+1}\ du = \frac{1}{3}\arctan(u) + C = \frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C$$

(d) $\displaystyle\int_0^1 e^{x+e^x}\,dx$

First notice that

$$\int_0^1 e^{x+e^x}\,dx = \int_0^1 e^x e^{e^x}\,dx.$$

Now, make the substitution $u = e^x$ with $du = e^x dx$ to get

$$\int_0^1 e^{x+e^x}\,dx = \int_0^1 e^x e^{e^x}\,dx = \int_1^e e^u\,du = e^u\big|_1^e = e^e - e$$

8. (a) Find the equation of the tangent line to $f(x) = x^3 \ln(x)$ at $x = 1$.

The point is $(1, f(1)) = (1, 0)$ and the slope is

$$f'(1) = 3x^2 \ln(x) + x^3 \frac{1}{x}\bigg|_{x=1} = 1.$$

So the equation of the tangent line is $y - 0 = 1(x - 1)$, or $y = x - 1$.

(b) Find the linear approximation to $g(x) = -x^3 \ln(x)$ at $x = 1$.

You find this the same way you find the tangent line (because they're the same thing). So, we have the point $(1, g(1)) = (1, 0)$ and the slope

$$g'(1) = -3x^2 \ln(x) - x^3 \frac{1}{x}\bigg|_{x=1} = -1.$$

Hence, the linear approximation is $L_1(x) - 0 = -1(x - 1)$ or

$$L_1(x) = -x + 1.$$

9. Give the expression for the right Riemann sum of n rectangles for the function $f(x) = x^3$ on the interval $[1, 4]$. Draw a picture of this when $n = 3$. Is this an under- or over-estimate?

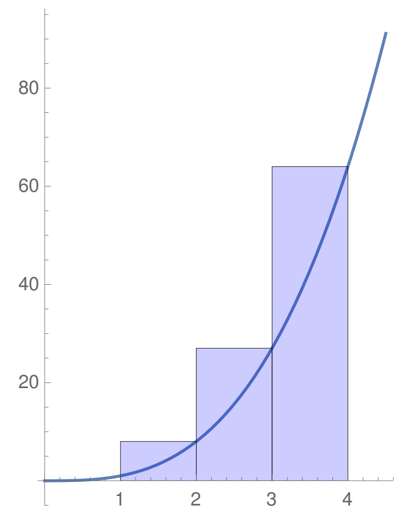
The right endpoint Riemann Sum for n rectangles is given by

$$\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}.$$

In our case $f(x) = x^3$, $a = 1$, and $b = 4$ so we get

$$\sum_{k=1}^{n} \left(1 + \frac{3k}{n}\right)^3 \frac{3}{n}.$$

For the picture, draw the graph of x^3 and then the right Riemann rectangles, as below.



From the picture we can see that the right Riemann sum is an over-estimate.