ESP Math 179

Worksheet 24 - solutions

Fall 2015

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2. (a) Give the nth midpoint Riemann sum of $f(x) = x^2 + x$ over $[0, 10]$. Do not simplify.

The interval we have is $[a, b] = [0, 10]$ and $f(x)$ is as given. Therefore the nth midpoint Riemann sum, called S_m, is

$$S_m = \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + (i-1) \cdot \frac{b-a}{n} + \frac{b-a}{2n}\right) \qquad \text{(using 1. in the same worksheet)}$$

$$= \sum_{i=1}^{n} \frac{10-0}{n} f\left(0 + (i-1) \cdot \frac{10-0}{n} + \frac{10-0}{2n}\right) \qquad \text{(letting } a = 0,\ b = 10\text{)}$$

$$= \sum_{i=1}^{n} \frac{10}{n} f\left(\frac{10i - 10}{n} + \frac{5}{n}\right) \qquad \text{(distributing and reducing)}$$

$$= \sum_{i=1}^{n} \frac{10}{n} f\left(\frac{10i - 5}{n}\right) \qquad \text{(combining fractions)}$$

$$= \sum_{i=1}^{n} \frac{10}{n} \left(\left(\frac{10i - 5}{n}\right)^2 + \frac{10i - 5}{n}\right). \qquad \text{(evaluating } f\text{)}$$

(b) Simplify your previous answer, using the following identites:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Here we continue from above, expanding and simplifying more. The calculations are started on the next page to make the steps clearer. We will use the above identities and the fact that

$$\sum_{i=1}^{n} 1 = n.$$

We continue to get

$$\begin{aligned}
S_m &= \sum_{i=1}^{n} \frac{10}{n}\left(\frac{100i^2-100i+25}{n^2}+\frac{10i-5}{n}\right)\\
&= \sum_{i=1}^{n}\left(\frac{1000i^2-1000i+250}{n^3}+\frac{100i-50}{n^2}\right)\\
&= \sum_{i=1}^{n}\left(\frac{1000i^2}{n^3}-\frac{1000i}{n^3}+\frac{250}{n^3}+\frac{100i}{n^2}-\frac{50}{n^2}\right)\\
&= \sum_{i=1}^{n}\frac{1000i^2}{n^3}-\sum_{i=1}^{n}\frac{1000i}{n^3}+\sum_{i=1}^{n}\frac{250}{n^3}+\sum_{i=1}^{n}\frac{100i}{n^2}-\sum_{i=1}^{n}\frac{50}{n^2}\\
&= \frac{1000}{n^3}\sum_{i=1}^{n} i^2-\frac{1000}{n^3}\sum_{i=1}^{n} i+\frac{250}{n^3}\sum_{i=1}^{n}1+\frac{100}{n^2}\sum_{i=1}^{n} i-\frac{50}{n^2}\sum_{i=1}^{n}1\\
&= \frac{1000}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{1000}{n^3}\cdot\frac{n(n+1)}{2}+\frac{250}{n^3}\cdot n+\frac{100}{n^2}\cdot\frac{n(n+1)}{2}-\frac{50}{n^2}\cdot n\\
&= \frac{1000(n^2+n)(2n+1)}{6n^3}-\frac{1000(n^2+n)}{2n^3}+\frac{250n}{n^3}+\frac{100(n^2+n)}{2n^2}-\frac{50n}{n^2}\\
&= \frac{1000(2n^3+n^2+2n^2+n)}{6n^3}-\frac{1000n^2+1000n}{2n^3}+\frac{250n}{n^3}+\frac{100n^2+100n}{2n^2}-\frac{50n}{n^2}\\
&= \frac{2000n^3+1000n^2+2000n^2+1000n}{6n^3}-\frac{1000n^2+1000n}{2n^3}+\frac{250n}{n^3}+\frac{100n^2+100n}{2n^2}-\frac{50n}{n^2}\\
&= \frac{1000n^2+500n+1000n+500}{3n^2}-\frac{500n+500}{n^2}+\frac{250}{n^2}+\frac{50n+50}{n}-\frac{50}{n}\\
&= \frac{1000}{3}+\frac{500}{3n}+\frac{1000}{3n}+\frac{500}{3n^2}-\frac{500}{n}+\frac{500}{n^2}+\frac{250}{n^2}+50+\frac{50}{n}-\frac{50}{n}\\
&= \frac{1}{n^2}\left(\frac{500}{3}+500+250\right)+\frac{1}{n}\left(\frac{500}{3}+\frac{1000}{3}-500+50-50\right)+\frac{1000}{3}+50.
\end{aligned}$$

It is fine to leave the answer in this form, without adding the numbers inside the parenthesis together, although you can add them if you wish.

(c) Take the limit of your previous answer, as $n \to \infty$.

We do as we're told, to find

$$\lim_{n\to\infty}\left[\frac{1}{n^2}\left(\frac{500}{3}+500+250\right)+\frac{1}{n}\left(\frac{500}{3}+\frac{1000}{3}-500+50-50\right)+\frac{1000}{3}+50\right]=\frac{1000}{3}+50.$$

(d) Compute the definite integral $\int_0^{10} x^2+x\,dx$ and compare it to your previous answer.

We quickly find that

$$\int_0^{10} x^2+x\,dx=\int_0^{10} x^2\,dx+\int_0^{10} x\,dx=\left(\frac{x^3}{3}+C_1\right)\Bigg|_{x=0}^{x=10}+\left(\frac{x^2}{2}+C_2\right)\Bigg|_{x=0}^{x=10}=\frac{1000}{3}+\frac{100}{2}=\frac{1000}{3}+50,$$

which is the same as we got above.