ESP Math 179 **Fall 2015**

Worksheet 9 - solutions

22 September 2015

2. **Intermediate value theorem:** A wildfire in the prairies starts at 5AM on Tuesday, and is spreading at a constant rate of 10 square miles per hour. Firefighting crews begin extinguishing the fire at 11AM at a constant rate of 16 square miles an hour.

   (a) Do the firefighters ever extinguish all the flames? If so, when? If no, why not?

   Yes, the firefighters extinguish the flames at 9PM. To see this, define a function

   $$f(x) = \text{area burning at time } x.$$

   Putting time in hours, from midnight, on the x-axis, and area, in square miles, on the y-axis, we can make a proper mathematical equations for this function. We know that f goes through the point $(5, 0)$ (representing 0 square miles burning at 5AM) and has a constant slope of 10 (representing 10 more square miles burning every hour) from 5AM to 11AM. At 11AM, 60 square miles are burning, so f goes through $(11, 60)$. From 11AM onward, the slope is -6, because while the burning is increasing by 10, the extinguishing is decreasing the burning area by 16 (so $10 - 16 = -6$). Applying the point-slope formula, we get:

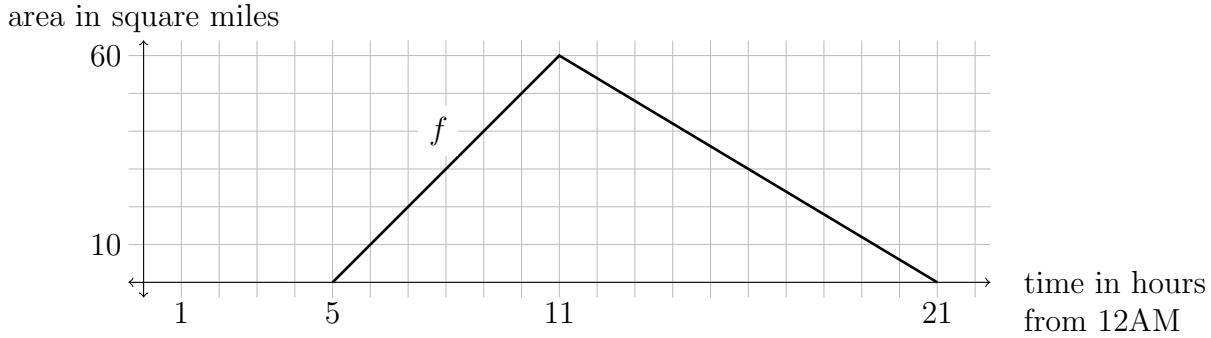
   $$\text{between 5AM and 11AM}: \quad \frac{y-0}{x-5} = 10 \quad \Longrightarrow \quad y = 10x - 50$$

   $$\text{after 11AM}: \quad \frac{y-60}{x-11} = -6 \quad \Longrightarrow \quad y = -6x + 126$$

   The firefighters have extinguished all the flames when $f(x) = 0$, for some $x > 11$ (because they only start extinguishing at 11). We know the form f has after 11AM, so we set f equal to 0 and solve:

   $$0 = -6x + 126 \quad \Longrightarrow \quad x = \frac{-126}{-6} = 21.$$

   Therefore the firefighters extinguish all the flames 21 hours after midnight, or at 9PM on Tuesday. We may even draw the function on a graph:



(b) Using the intermediate value theorem, show that there is a time when the area burning is equal to the area already extinguished by the firefighters.

Define another function $g(x)$ by

$$g(x) = \text{area extinguished at time } x.$$

As above, let's make a mathematical expression for this function. We know g goes through $(11, 0)$ (representing 0 square miles extinguished at 11AM) and has a constant slope of 16 after 11AM. With point-slope:

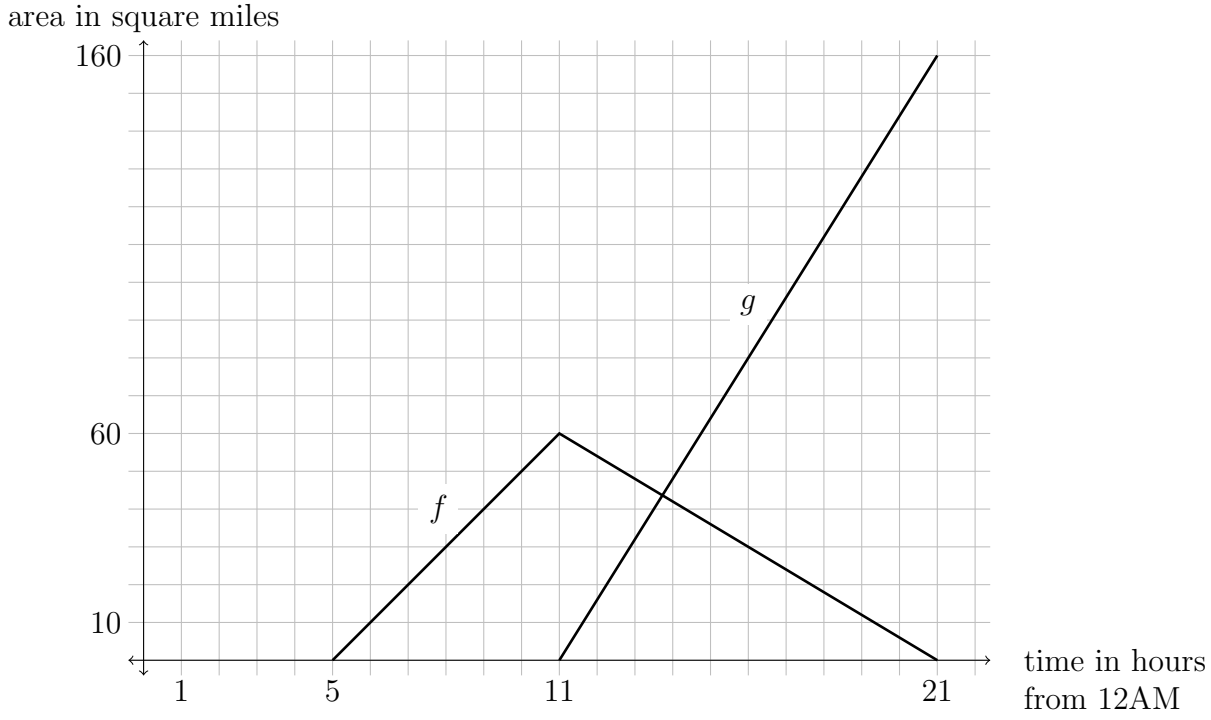
$$\text{after 11AM}: \quad \frac{y-0}{x-11} = 16 \quad \Longrightarrow \quad y = 16x - 176$$

The question asks for a time when the area burning is equal to the area extinguished. This is the same as asking for which x is $f(x) = g(x)$, or when $f(x) - g(x) = 0$. Note that both f and g are continuous on $[11, 21]$ (representing time between 11AM and 9PM), so the function $h(x) = f(x) - g(x)$ is also continuous on $[11, 21]$. Then

$$h(11) = f(11) - g(11) = 60 - 0 = 60,$$
$$h(21) = f(21) - g(21) = 0 - 160 = -160.$$

Since $h(11) > 0$ and $h(21) < 0$, by the intermediate value theorem there is some $x \in (11, 21)$ with $h(x) = 0$. That is, there is some time between 11AM and 9PM when the area burning is equal to the area extinguished.

We may even draw the function on the graph, together with f:



The time when the area burning is equal to the area extinguished is the point where the two functions intersect, somewhere between 1PM and 2PM.