Assignment 9

Introduction to Linear Algebra

Material from Lectures 16 and 17

Due Thursday, March 16, 2023

16.1 (✠2.17)For each of the following "definitions", show that each cannot be an inner product.

(a) For $A, B \in \mathcal{M}_{n\times n}$, let $\langle A, B\rangle = \text{trace}(A + B)$

(b) For $f, g \in C[0, 1]$, let $\langle f, g\rangle = \left|\frac{df}{dx}\frac{dg}{dx}\right|$

(c) For $a, b \in \mathbf{R}$, let $\langle a, b\rangle = a^2 + b^2$

16.4 (✠2.18)Let $P(\mathbf{R})$ be the vector space of all polynomials $\mathbf{R} \to \mathbf{R}$, with scalar multiplication and polynomial addition defined as you would expect. You may assume that the following is an inner product on $P(\mathbf{R})$:

$$\langle p(x), q(x)\rangle = \int_0^\infty p(x)q(x)e^{-x}\,dx.$$

(a) Check that $p(x) = 2x - 1$ and $q(x) = x + 3$ are not orthogonal to each other.

(b) Using the Gram–Schmidt process on $p(x)$ and $q(x)$ as in part 1., find a polynomial $r(x) \in P(\mathbf{R})$ that is orthogonal to $p(x)$. Give your answer as $r(x) = ax + b$, for $a, b \in \mathbf{Z}$.

17.4 (✠2.20) Consider the three orthogonal vectors

$$\mathbf{x} = \begin{bmatrix}1\\0\\3\end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix}3\\0\\-1\end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix}0\\-2\\0\end{bmatrix}.$$

Let $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation for which

$$f(\mathbf{x}) = \begin{bmatrix}1\\1\\0\end{bmatrix}, \qquad f(\mathbf{y}) = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}, \qquad f(\mathbf{z}) = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Construct the 3×3 matrix for f.

17.5 (✠2.20, 2.21) Let V be the vector space of polynomials in two variables x and y of degree at most 2. This space has dimension 6, and has basis with basis $1, x, y, x^2, y^2, xy$. Let $L\colon V \to V$ be the linear transformation defined by $L(f(x, y)) = f(x - y, y - x)$.

(a) Find the matrix of L using the basis specified.

(b) Find a basis for the image and kernel of L.