Assignment 9 - Solutions

Introduction to Linear Algebra

Material from Lectures 16 and 17

Due Thursday, March 16, 2023

16.1 (✠2.17)For each of the following "definitions", show that each cannot be an inner product.

(a) For $A, B \in \mathcal{M}_{n\times n}$, let $\langle A, B\rangle = \text{trace}(A + B)$

(b) For $f, g \in C[0, 1]$, let $\langle f, g\rangle = \left|\frac{df}{dx}\frac{dg}{dx}\right|$

(c) For $a, b \in \mathbf{R}$, let $\langle a, b\rangle = a^2 + b^2$

For each we give an example where it fails.

(a) For $A = B = -I$, the trace is negative, but the inner product must always be greater than or equal to zero. That is, this "definition" of inner product does not work, as it is not positive definite.

(b) For $f = 1$, the derivative is zero, but f is not zero, and the inner product $\langle f, f\rangle$ evaluates to zero only when f is zero. That is, this "definition" of inner product does not work, as it is not positive definite.

(c) For $a = 2, b = 4$, by multiplicativity we should have $\langle 2, 4\rangle = \langle 2, 2 \cdot 2\rangle = 2\langle 2, 2\rangle$. But instead we have $\langle 2, 4\rangle = 20$ and $2\langle 2, 2\rangle = 16$. That is, this "definition" of inner product does not work, as it does not satisfy the multiplicative property.

There are other counterexamples.

16.4 (✠2.18)Let $P(\mathbf{R})$ be the vector space of all polynomials $\mathbf{R} \to \mathbf{R}$, with scalar multiplication and polynomial addition defined as you would expect. You may assume that the following is an inner product on $P(\mathbf{R})$:

$$\langle p(x), q(x) \rangle = \int_0^\infty p(x)q(x)e^{-x}\, dx.$$

(a) Check that $p(x) = 2x - 1$ and $q(x) = x + 3$ are not orthogonal to each other.

(b) Using the Gram–Schmidt process on $p(x)$ and $q(x)$ as in part 1., find a polynomial $r(x) \in P(\mathbf{R})$ that is orthogonal to $p(x)$. Give your answer as $r(x) = ax + b$, for $a, b \in \mathbf{Z}$.

(a) If the two given functions were orthogonal, we should get $\langle 2x - 1, x + 3 \rangle = 0$. However, we find that

$$\begin{aligned}
\langle 2x - 1, x + 3 \rangle &= \int_0^\infty (2x - 1)(x + 3)e^{-x}\, dx \\
&= \int_0^\infty (2x^2 + 5x - 3)e^{-x}\, dx \\
&= 2\int_0^\infty x^2 e^{-x}\, dx + 5\int_0^\infty xe^{-x}\, dx - 3\int_0^\infty e^{-x}\, dx \\
&= 2\Big((-x^2 - 2x - 2)e^{-x}\Big)_{x=0}^{x=\infty} + 5\Big((-x - 1)e^{-x}\Big)_{x=0}^{x=\infty} - 3\Big(-e^{-x}\Big)_{x=0}^{x=\infty} \\
&= 2 \cdot 2 + 5 \cdot 1 - 3 \cdot 1 \\
&= 6.
\end{aligned}$$

Here we used integration by parts. The value of this integral is not zero, so the functions are not orthogonal.

(b) Computing the second vector in the Gram–Schmidt process is done by

$$\begin{aligned}
q(x) - \mathrm{proj}_{p(x)}(q(x)) &= x + 3 - \frac{\langle 2x - 1, x + 3 \rangle}{\langle 2x - 1, 2x - 1 \rangle}(2x - 1) \\
&= x + 3 - \frac{6}{5}(2x - 1) \\
&= \frac{1}{5}(5x + 15 - 12x + 6) \\
&= \frac{1}{5}(-7x + 21).
\end{aligned}$$

This function and all its multiples, such as $r(x) = 7x - 21$, are orthogonal to $p(x)$.

17.4 (✠2.20) Consider the three orthogonal vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}.$$

Let $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation for which

$$f(\mathbf{x}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad f(\mathbf{y}) = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \qquad f(\mathbf{z}) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Construct the 3×3 matrix for f.

We follow the proof of Theorem 17.12 to get a matrix for this linear transformation. First we compute what f does to the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We construct these from the given $\mathbf{x}, \mathbf{y}, \mathbf{z}$ as

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \frac{3}{10}\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{3}{10}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

Now we evaluate f on these vectors and apply the two properties of linearity:

$$f\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = f\left(\frac{1}{10}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \frac{3}{10}\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}\right) = \frac{1}{10}f\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \frac{3}{10}f\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{10}\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -1/5 \\ -3/10 \end{bmatrix},$$

$$f\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = f\left(-\frac{1}{2}\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}\right) = -\frac{1}{2}f\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix},$$

$$f\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = f\left(\frac{3}{10}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}\right) = \frac{3}{10}f\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \frac{3}{10}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 2/5 \\ 1/10 \end{bmatrix}.$$

Hence the matrix of f is

$$f\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/5 & 0 & 2/5 \\ -1/5 & -1/2 & 2/5 \\ -3/10 & 1/2 & 1/10 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -2 & 0 & 4 \\ -2 & -5 & 4 \\ -3 & 5 & 1 \end{bmatrix}.$$

17.5 (✠2.20, 2.21) Let V be the vector space of polynomials in two variables x and y of degree at most 2. This space has dimension 6, and has basis with basis $1, x, y, x^2, y^2, xy$. Let $L\colon V \to V$ be the linear transformation defined by $L(f(x,y)) = f(x-y, y-x)$.

(a) Find the matrix of L using the basis specified.

(b) Find a basis for the image and kernel of L.

(a) We construct the matrix by unerstanding what it does to the basis vectors.

$$\begin{aligned} L(1) &= 1 \\ L(x) &= x - y \\ L(y) &= y - x \\ L(x^2) &= (x-y)^2 = x^2 + y^2 - 2xy \\ L(y^2) &= (y-x)^2 = x^2 + y^2 - 2xy \\ L(xy) &= (x-y)(y-x) = -x^2 - y^2 - 2xy \end{aligned}$$

This immediately gives us the matrix of L, where the rows and columns respresent the basis elements in the order $1, x, y, x^2, y^2, xy$:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -2 & -2 & -2 \end{bmatrix}.$$

(b) The image is the column space, and the kernel is the nullspace. Since

$$\text{col}(L) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -2 \end{bmatrix} \right\},$$

it follows that $\text{im}(L) = \text{span}\{1, x-y, x^2+y^2-2xy, -x^2-y^2-2xy\}$. For the nullspace, we quickly bring L to row reduced echelon form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -2 & -2 & 0 \end{bmatrix} \xrightarrow{G.E.} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The row reduced form of L has two free columns, so

$$\text{null}(L) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\},$$

meaning that $\ker(L) = \text{span}\{x + y, -x^2 + xy\}$.