Assignment 8

Introduction to Linear Algebra

Material from Lectures 14 and 15
Due Thursday, March 9, 2023

14.2 (✠2.13) Let $A \in \mathcal{M}_{m\times n}$.

(a) Suppose that $\mathbf{x} \in \text{null}(A)$. Show that $\mathbf{x} \in \text{null}(A^TA)$ as well

(b) Suppose that $\mathbf{y} \in \text{null}(A^TA)$. Show that $\mathbf{y} \in \text{null}(A)$ as well.

(c) The above two points imply that $\text{null}(A) = \text{null}(A^TA)$. In the case that the columns of A are linearly independent, use this fact to show that A^TA has full rank.

14.3 (✠2.14) Consider the two points $p_1 = (1, 0)$, $p_2 = (2, 1)$ in $\mathbf{R}^2$.

(a) Let $m \in \mathbf{R}$. Find a point $p_3 \in \mathbf{R}^2$ so that the least squares approximation to $\{p_1, p_2, p_3\}$ has slope m.

(b) Let $a \geqslant 3 \in \mathbf{R}$. Find points $p_3, p_4 \in \mathbf{R}^2$ so that the degree 2 least squares approximation to $\{p_1, p_2, p_3, p_4\}$ has its vertex on the line $x = a$.

15.2 (✠2.15) Apply the Gram-Schmidt process to the vectors $\begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\1\end{bmatrix}$.

15.4 (✠2.16) Let V be the vector space of polynomials of degree at most 3 with domain $[0, 1]$. The "dot product" on V is defined by $f \bullet g = \int_0^1 f(x)g(x)\,dx$, which helps to define length and angle. You may assume that $\{1, x, x^2, x^3\}$ is a basis for V.

(a) Is the given basis orthogonal? Find the lengths of the elements in the basis.

(b) Are the two functions $2x, x^2 - 1$ linearly independent in V? Are they orthogonal?

(c) Extend $\{2x, x^2 - 1\}$ to an orthonormal basis of V.