Assignment 7 - Solutions

Introduction to Linear Algebra

Material from Lectures 12 and 13
Due Thursday, March 2, 2023

12.1 (✠2.09) Let $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ be a symmetric 4×4 matrix.

(a) Find which pairs of columns of A are orthogonal to each other.

(b) Give the nullspace of A as a span of the special solutions to $A\mathbf{x} = 0$.

(c) Show that the column space of A is orthogonal to the nullspace of A.

(d) Explain why for any symmetric matrix (not just the one given), its column space is orthogonal to its nullspace.

Let $A = [\,\mathbf{c}_1\ \mathbf{c}_1\ \mathbf{c}_3\ \mathbf{c}_4\,]$, with the columns $\mathbf{c}_i$ as in the matrix A above.

(a) Column 1 is orthogonal to columns 3 and 4, as

$$\mathbf{c}_1 \bullet \mathbf{c}_3 = \mathbf{c}_1 \bullet \mathbf{c}_4 = 0.$$

Column 2 is also orthogonal to columns 3 and 4, as

$$\mathbf{c}_2 \bullet \mathbf{c}_3 = \mathbf{c}_2 \bullet \mathbf{c}_4 = 0.$$

Column 3 is orthogonal to columns 1 and 2, as described above, and also to column 4, as $\mathbf{c}_3 \bullet \mathbf{c}_4 = 0$. Column 4 is orthogonal to columns 1, 2, 3.

(b) Row reduction will give us the nullspace, and we see that

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{adding rows}} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{\text{multiplying rows}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Columns 1,3,4 are pivot columns and column 2 is a free column. Therefore

$$\text{null}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right).$$

(c) Note that the only vector $\mathbf{n} = \left[\begin{smallmatrix} 1 \\ 1 \\ 0 \\ 0 \end{smallmatrix}\right]$ in the spanning set of null(A) is orthogonal to every column of A:

$$\mathbf{n} \bullet \mathbf{c}_1 = \mathbf{n} \bullet \mathbf{c}_2 = \mathbf{n} \bullet \mathbf{c}_3 = \mathbf{n} \bullet \mathbf{c}_4 = 0.$$

Any vector $\mathbf{v} = a\mathbf{c}_1 + b\mathbf{c}_2 + c\mathbf{c}_3 + d\mathbf{c}_4 \in \mathrm{col}(A)$ will then be orthogonal to $\mathbf{n}$, as

$$\begin{aligned}\mathbf{n} \bullet \mathbf{v} &= \mathbf{n} \bullet (a\mathbf{c}_1 + b\mathbf{c}_2 + c\mathbf{c}_3 + d\mathbf{c}_4) \\ &= a\mathbf{n} \bullet \mathbf{c}_1 + b\mathbf{n} \bullet \mathbf{c}_2 + c\mathbf{n} \bullet \mathbf{c}_3 + d\mathbf{n} \bullet \mathbf{c}_4 \\ &= a0 + b0 + c0 + d0 \\ &= 0.\end{aligned}$$

Since every vector in $\mathrm{null}(A)$ is a multiple of $\mathbf{n}$, it follows that every vector in $\mathrm{null}(A)$ will be orthogonal to every vector in $\mathrm{col}(A)$. That is, $\mathrm{null}(A)$ is orhogonal to $\mathrm{col}(A)$.

(d) By Example 12.13 in the lecture notes, the column space of A is orthogonal to the left nullspace of A. Since A is symmetric, we have that $A = A^T$, and so the left nullspace is the same as the nullspace, that is, $\mathrm{null}(A^T) = \mathrm{null}(A)$. Hence for any symmetric matrix A, the column space of A is orthogonal to the nullspace of A.

12.6 (✠2.10) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ -1 & -2 \\ 1 & 3 \end{bmatrix}$.

(a) What are the dimensions of $\mathrm{col}(A)$ and $\mathrm{col}(B)$? Only using dimensions, explain why $\mathrm{col}(A) \neq \mathrm{col}(B)^\perp$.

(b) Find a vector that is both in $\mathrm{col}(A)$ and $\mathrm{col}(B)$.
Hint: Row reduce the block matrix $\begin{bmatrix} A & B \end{bmatrix}$.

(a) A matrix with two columns has a column space of dimension 0, 1, or 2. The dimension is 0 if both columns are the zero column (which is clearly not the case for A and B). The dimension is 1 if one column is a multiple of the other. This is also not the case, as the ratios are different among rows:

$$\frac{1}{2} \neq \frac{1}{3} \neq \frac{2}{5} \quad \text{for } A \text{, and} \quad \frac{-1}{2} \neq \frac{-1}{-2} \neq \frac{1}{3} \quad \text{for } B.$$

Therefore $\dim(\mathrm{col}(A)) = \dim(\mathrm{col}(B)) = 2$.

Since $\mathrm{col}(B) \subseteq \mathbf{R}^3$, and $\dim(\mathbf{R}^3) = 3$, for whichever space $V = \mathrm{col}(B)^\perp$ that is the orthogonal complement of $\mathrm{col}(B)$, by Definition 12.9 it must be that

$$\dim(\mathrm{col}(B)) + \dim(V) = \dim(\mathbf{R}^3) = 3.$$

Since $\dim(\mathrm{col}(B)) = 2$, it must be that $\dim(V) = 1$. Since $\dim(\mathrm{col}(A)) = 2 \neq 1$, it follows that $\mathrm{col}(A) \neq V$, that is, $\mathrm{col}(A) \neq \mathrm{col}(B)^\perp$.

(b) There are many such vectors. We follow the hint and row reduce

$$\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 1 & 3 & -1 & -2 \\ 2 & 5 & 1 & 3 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The result of this row reduction implies that column 4 of $[A\ B]$ is in the span of the first three columns of $[A\ B]$, with the relationship

$$11 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}.$$

Rearranging, we get that

$$11 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = -\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix},$$

the sum of each side being the same vector $\mathbf{v} = \left[\begin{smallmatrix} 3 \\ -1 \\ 2 \end{smallmatrix}\right]$. Since the left side is a linear combination of the columns of A, $\mathbf{v}$ is in the column space of A. Similarly, since the right side is a linear combination of the columns of B, $\mathbf{v}$ is in the column space of B. Therefore $\mathbf{v} \in \mathrm{col}(A)$ and $\mathbf{v} \in \mathrm{col}(B)$ as desired.

13.1 (✠2.11) This question is about repeated projections.

(a) Show that projecting twice onto a line is the same as projecting once.

(b) Show that projecting twice onto a subspace is the same as projecting once.

Hint: Use the projection matrices P from Equation (4) and Definition 13.6, and show that $P^2 = P$.

(c) Let $R_\theta \in \mathcal{M}_{2\times 2}$ be the rotation matrix from Example 12.7. For which $\theta \in [0, 2\pi)$ is R_θ a projection matrix? Justify your answer.

(a) From Lecture 13, after Definition 13.2, the matrix for projecting a vector $\mathbf{u}$ onto the line given by vector $\mathbf{v}$ is

$$P = \frac{1}{\mathbf{v} \bullet \mathbf{v}} \cdot \mathbf{v}\mathbf{v}^T,$$

so the projection of $\mathbf{u}$ onto the line defined by $\mathbf{v}$ is $P\mathbf{u}$, and projecting twice would be $PP\mathbf{u}$. Observe that

$$\begin{aligned} PP &= \left(\frac{1}{\mathbf{v} \bullet \mathbf{v}} \cdot \mathbf{v}\mathbf{v}^T\right) \cdot \left(\frac{1}{\mathbf{v} \bullet \mathbf{v}} \cdot \mathbf{v}\mathbf{v}^T\right) \\ &= \frac{(\mathbf{v}\mathbf{v}^T)(\mathbf{v}\mathbf{v}^T)}{(\mathbf{v} \bullet \mathbf{v})(\mathbf{v} \bullet \mathbf{v})} \\ &= \frac{\mathbf{v}(\mathbf{v}^T\mathbf{v})\mathbf{v}^T}{(\mathbf{v} \bullet \mathbf{v})(\mathbf{v} \bullet \mathbf{v})} \\ &= \frac{\mathbf{v}(\mathbf{v} \bullet \mathbf{v})\mathbf{v}^T}{(\mathbf{v} \bullet \mathbf{v})(\mathbf{v} \bullet \mathbf{v})} \\ &= \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v} \bullet \mathbf{v}} \\ &= P, \end{aligned}$$

so $PP\mathbf{u} = P\mathbf{u}$. That is, projecting onto a line once is the same as projecting onto a line twice.

(b) Recall from Lecture 13, Definition 13.6, that the matrix for projecting a vector $\mathbf{u} \in \mathbf{R}^n$ onto a subspace of $\mathbf{R}^n$ whose basis vectors are the columns of A is

$$P = A(A^TA)^{-1}A^T,$$

so the projection of $\mathbf{u}$ onto the column space of A is $P\mathbf{u}$, and projecting twice would be $PP\mathbf{u}$. Observe that

$$\begin{aligned} PP &= \left(A(A^TA)^{-1}A^T\right)\left(A(A^TA)^{-1}A^T\right) \\ &= A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T \\ &= AI(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}A^T \\ &= P, \end{aligned}$$

so $PP\mathbf{u} = P\mathbf{u}$. That is, projecting onto a subspace once is the same as projecting onto a subspace twice.

(c) By the definition of R_θ, we know that $R_\theta R_\theta = R_{2\theta}$. That is, rotating by θ twice is the same as rotating by 2θ once. By part (a), for R_θ to be a projection matrix, we must have that $R_\theta R_\theta = R_\theta$, that is, we need $2\theta = \theta$. This is only true for $\theta = 0$.

13.5 (✠2.12) Let $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ 2 & 1 & 1 & -2 \\ -1 & 0 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 1 & -1 \\ 0 & -1 \\ 0 & 2 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

(a) Compute the projection of $\mathbf{v}$ onto $\mathrm{col}(A)$ and $\mathrm{col}(A)^\perp$. What is the angle between the two projections?

(b) Compute the projection of $\mathrm{col}(B)$ onto $\mathrm{col}(A)$.

(a) To compute the projection, we need a basis for $\mathrm{col}(A)$ and $\mathrm{col}(A)^\perp$. By row reduction, A has three pivots, and rows 1,2,4 are linearly independent. Define the matrix

$$C = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix},$$

which has lin. indep. columns. Projecting onto $\mathrm{col}(A) = \mathrm{col}(C)$ is done by

$$P = C(C^TC)^{-1}C^T = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -2 \\ 0 & 1 & 3 & -1 & -1 \\ 0 & 1 & -1 & 3 & -1 \\ 0 & -1 & -1 & -1 & 1 \end{bmatrix}.$$

Using the formulas in Definition 13.6 and Remark 13.7, we see that

$$\mathrm{proj}_{\mathrm{col}(A)}(\mathbf{v}) = P\mathbf{v} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -2 \\ 0 & 1 & 3 & -1 & -1 \\ 0 & 1 & -1 & 3 & -1 \\ 0 & -1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ -1 \\ 0 \\ 1/2 \end{bmatrix},$$

$$\mathrm{proj}_{\mathrm{col}(A)^\perp}(\mathbf{v}) = (I-P)\mathbf{v} = \mathbf{v} - P\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1/2 \\ -1 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}.$$

The angle between these two projections is $\pi/2$ as the dot product is zero.

(b) A subspace is the span of its basis vectors, and we know the basis vectors for $\mathrm{col}(B)$ are the columns of B (as they are not multiples of each other). The projection of a subspace is the span of the projection of its basis vectors, so we just need to compute the projection of the columns of B onto $\mathrm{col}(A)$. We already have the projection matrix P from part (a), with which we find

$$P\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/4 \\ 5/4 \\ 1/4 \\ -3/4 \end{bmatrix}, \qquad P\begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3/4 \\ -3/4 \\ -3/4 \\ 3/4 \end{bmatrix},$$

and the span of these vectors is the projection of $\mathrm{col}(B)$ onto $\mathrm{col}(A)$.