Assignment 6 - Solutions

Introduction to Linear Algebra

Material from Lectures 10 and 11

Due Thursday, February 16, 2023

10.1 (✠2.03) Find all sets of size 3 from the vectors below that are linearly independent:

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}2\\0\\1\end{bmatrix}, \quad \begin{bmatrix}2\\0\\2\end{bmatrix}, \quad \begin{bmatrix}3\\-1\\3\end{bmatrix}.$$

Choosing 3 vectors from 5 gives $\binom{5}{3} = 10$ choices, so there are at most 10 such sets. We see that any set containing $\left[\begin{smallmatrix}2\\0\\2\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right]$ is not linearly independent (there are 3 such sets), because

$$\begin{bmatrix}2\\0\\2\end{bmatrix} = 2\begin{bmatrix}1\\0\\1\end{bmatrix}.$$

Similarly, the set containing $\left[\begin{smallmatrix}3\\-1\\3\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]$ is not linearly independent, because

$$\begin{bmatrix}3\\-1\\3\end{bmatrix} = 3\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

For a similar reason, the set containing $\left[\begin{smallmatrix}3\\-1\\3\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}2\\0\\2\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]$ is not linearly independent, because

$$\begin{bmatrix}3\\-1\\3\end{bmatrix} = \frac{3}{2}\begin{bmatrix}2\\0\\2\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

The mentioned sets do not overlap (that is, never satisfy more than one of the stated relationships). This leaves five linearly independent sets of three vectors each. We check that row reduction on matrices containg them as columns does indeed produce three pivots.

$$\left\{\left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}2\\0\\1\end{smallmatrix}\right]\right\} \xrightarrow{\text{as a matrix}} \left[\begin{smallmatrix}1&0&2\\0&1&0\\1&0&1\end{smallmatrix}\right] \xrightarrow{\text{Gaussian elimination}} \left[\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right]$$

$$\left\{\left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}2\\0\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}3\\-1\\3\end{smallmatrix}\right]\right\} \xrightarrow{\text{as a matrix}} \left[\begin{smallmatrix}1&2&3\\0&0&-1\\1&1&3\end{smallmatrix}\right] \xrightarrow{\text{Gaussian elimination}} \left[\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right]$$

$$\left\{\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}2\\0\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}2\\0\\2\end{smallmatrix}\right]\right\} \xrightarrow{\text{as a matrix}} \left[\begin{smallmatrix}0&2&2\\1&0&0\\0&1&2\end{smallmatrix}\right] \xrightarrow{\text{Gaussian elimination}} \left[\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right]$$

$$\left\{\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}2\\0\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}3\\-1\\3\end{smallmatrix}\right]\right\} \xrightarrow{\text{as a matrix}} \left[\begin{smallmatrix}0&2&3\\1&0&-1\\0&1&3\end{smallmatrix}\right] \xrightarrow{\text{Gaussian elimination}} \left[\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right]$$

$$\left\{\left[\begin{smallmatrix}2\\0\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}2\\0\\2\end{smallmatrix}\right],\left[\begin{smallmatrix}3\\-1\\3\end{smallmatrix}\right]\right\} \xrightarrow{\text{as a matrix}} \left[\begin{smallmatrix}2&2&3\\0&0&-1\\1&2&3\end{smallmatrix}\right] \xrightarrow{\text{Gaussian elimination}} \left[\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right]$$

These are the five necessary sets. These computations may be done with computer assistance.

10.5 (✠2.04) This question is about expressing vectors in different bases.

(a) Express the vector $\begin{bmatrix} 3 \\ -2 \\ -8 \end{bmatrix}$ in the basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(b) There are two bases $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of a vector space V, with the following relations:

$$\mathbf{b}_1 = \mathbf{a}_1 + \mathbf{a}_2, \qquad \mathbf{b}_2 = \mathbf{a}_2 + \mathbf{a}_3, \qquad \mathbf{b}_3 = \mathbf{a}_1 + \mathbf{a}_3.$$

If you know that $\mathbf{v} = 3\mathbf{a}_1 - 2\mathbf{a}_2 - 8\mathbf{a}_3$, express $\mathbf{v}$ as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$.

(a) To express the given vector in the given basis, we must solve the equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ -8 \end{bmatrix}.$$

With the help of a computer, we find the solution to be $\mathbf{x} = \begin{bmatrix} 9/2 \\ -13/2 \\ -3/2 \end{bmatrix}$. Therefore

$$\begin{bmatrix} 3 \\ -2 \\ -8 \end{bmatrix} = \frac{9}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{13}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Notice that the coefficients of $\mathbf{v}$ in the given equation are the same as in the given vector for part (a). So we multiply the solved matrix equation from part (a) on both sides by the row vector $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ to get

$$\begin{aligned}
\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9/2 \\ -13/2 \\ -3/2 \end{bmatrix} &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -8 \end{bmatrix} \\
\begin{bmatrix} \mathbf{a}_1 + \mathbf{a}_2 & \mathbf{a}_2 + \mathbf{a}_3 & \mathbf{a}_1 + \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} 9/2 \\ -13/2 \\ -3/2 \end{bmatrix} &= 3\mathbf{a}_1 - 2\mathbf{a}_2 - 8\mathbf{a}_3 \\
\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} \begin{bmatrix} 9/2 \\ -13/2 \\ -3/2 \end{bmatrix} &= \mathbf{v} \\
\frac{9}{2}\mathbf{b}_1 - \frac{13}{2}\mathbf{b}_2 - \frac{3}{2}\mathbf{b}_3 &= \mathbf{v}.
\end{aligned}$$

This gives $\mathbf{v}$ as a linear combination of $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$. There are other ways to solve this, by back substitution, but since we have already done all the necessary work in part (a), we do not need to do more work.

11.1 (✠2.07) Consider the plane $P = \{(x, y, z) \in \mathbf{R}^3 \ : \ 2x - 4y - 5z = 0\}$, which is a subspace of $\mathbf{R}^3$.

(a) Find a vector $\mathbf{n}$ normal to the plane P. That is, find $\mathbf{n} \in \mathbf{R}^3$ so that $\mathbf{n} \bullet \mathbf{v} = 0$, for $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ a solution to $2x - 4y - 5z = 0$.

(b) Considering the vector $\mathbf{n}$ as a 1×3 matrix A, the nullspace of A is precisely all points in the plane P. Find this nullspace, and express it as a span.

(c) What is a basis for P?

(a) The normal vector to a plane is already given in its equation. The vector (x, y, z) is in P if $2x - 4y - 5z = 0$, or equivalently, if

$$\begin{bmatrix} 2 \\ -4 \\ -5 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

Hence $\mathbf{n} = \begin{bmatrix} 2 \\ -4 \\ -5 \end{bmatrix}$ is normal to P.

(b) As prompted, we row reduce $\mathbf{n}$ as a row vector, or a 1×3 matrix A:

$$A = \begin{bmatrix} 2 & -4 & -5 \end{bmatrix} \xrightarrow{\text{Gaussian elimination}} \begin{bmatrix} 1 & -2 & -\frac{5}{2} \end{bmatrix} = R.$$

There are two free columns, so there are two special solutions. Using the algorithm for constructing the nullspace, we find them quickly to be

$$\mathbf{s}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{s}_2 = \begin{bmatrix} \frac{5}{2} \\ 0 \\ 1 \end{bmatrix}.$$

That is, $\text{null}(A) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{5}{2} \\ 0 \\ 1 \end{bmatrix} \right)$.

(c) A plane is 2-dimensional, so it should have two elements in the basis. Note that the defining equation may be expressed as

$$\begin{bmatrix} 2 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In other words, the points (x, y, z) that are in the plane P are precisely the points in the nullspace of $\begin{bmatrix} 2 & -4 & -5 \end{bmatrix}$. We found this nullspace in part (b), hence $\{\mathbf{s}_1, \mathbf{s}_2\}$ is a basis for P. Knowing that $\dim(P) = 2$ and that $P = \text{span}(\mathbf{s}_1, \mathbf{s}_2)$, we do not need to check that $\mathbf{s}_1, \mathbf{s}_2$ are linearly independent (since at least two vectors are needed to span a 2-dimensional space), as every basis of P must have 2 vectors. Hence $\{\mathbf{s}_1, \mathbf{s}_2\}$ is indeed a basis for P.

11.2 (✠2.08) For $a, b, c \in \mathbf{R}$, consider the matrix

$$A = \begin{bmatrix} 0 & 1 & a & 0 & a & 0 \\ 0 & 0 & 1 & b & 0 & b \\ 0 & 0 & 0 & 1 & c & c \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the column space, nullspace, row space, and left nullspace of A.

(b) Do the dimensions of the four fundamental spaces change if all of a, b, c are zero?

(a) To find bases for these spaces, we have to row reduce the matrix and its transpose:

$$\text{rref}(A) = \begin{bmatrix} 0 & 1 & 0 & 0 & a+abc & abc-ab \\ 0 & 0 & 1 & 0 & -bc & b-bc \\ 0 & 0 & 0 & 1 & c & c \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \text{rref}(A^T) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the row-reduced form of A, we see columns 2,3,4 of A are pivot columns, and columns 1,5,6 of A are free columns. Therefore

$$\text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b \\ 1 \\ 0 \end{bmatrix} \right), \qquad \text{null}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -a-abc \\ bc \\ -c \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ ab-abc \\ bc-b \\ -c \\ 0 \\ 1 \end{bmatrix} \right).$$

The vectors given in the arguments of the span form a basis. This follows as linear independence of the pivot columns is guaranteed by the definition of a pivot (zeros below it), and linear independence of the special solutions is guaranteed by the algorithm to construct them (1's and 0's in the rows of the free column indices).

From the row-reduced form of A^T, we see columns 1,2,3 of A^T are pivot columns, and columns 4 of A^T is a free column. Therefore

$$\text{row}(A) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ a \\ 0 \\ a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ b \\ 0 \\ b \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ c \\ c \end{bmatrix} \right), \qquad \text{null}(A^T) = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

The vectors given in the arguments of the span form a basis for the same reasons as above.

(b) Dimensions of these spaces would change if one of the basis vectors had only $0, a, b, c$, and no other numbers, but there are no such vectors among the ones above. Therefore the dimensions of these four fundamental subpaces do not change if any / all of a, b, c are zero.