Assignment 5

Introduction to Linear Algebra

Material from Lectures 8 and 9
Due Thursday, February 9, 2023

8.2 (✠1.14) Find the complete solution to $A\mathbf{x} = \mathbf{b}$, for

$$A = \begin{bmatrix} 3 & 0 & -9 & -3 & 0 \\ 6 & 0 & -21 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 9 \\ -1 \\ 0 \end{bmatrix}.$$

8.5 (✠1.15) Consider the vectors $\mathbf{a} = \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b \\ b \\ b \\ b \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, for $a, b \in \mathbf{R}$.

(a) What will be the rank of the following 4×4 matrices:

i. $\mathbf{a}\mathbf{u}^T$ ii. $\mathbf{b}\mathbf{v}^T$ iii. $\mathbf{a}\mathbf{u}^T + \mathbf{b}\mathbf{v}^T$

Your answers should depend on a and b.

(b) Explain why the rank of $\mathbf{x}\mathbf{y}^T$, for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, can never be greater than 1.

9.2 (✠4.02) Prove all the claims of Proposition 9.5, for $z = x + yi, w = a + bi \in \mathbf{C}$:

(a) $\overline{z + w} = \overline{z} + \overline{w}$

(b) $\overline{zw} = \overline{z}\,\overline{w}$

(c) $\overline{\overline{z}} = z$

(d) $z + \overline{z} = 2x$

(e) $z - \overline{z} = 2yi$

(f) $z^{-1} = \overline{z}/|z|^2$ for $z \neq 0$

(g) $|z| = 0$ iff $z = 0$

(h) $|\overline{z}| = |z|$

(i) $|zw| = |z||w|$

(j) $|z + w| \leqslant |z| + |w|$

9.3 (✠4.01) This question is about proving Euler's formula $\cos(\theta) + i\sin(\theta) = e^{i\theta}$.

(a) Take the derivative of $f(\theta) = (\cos(\theta) + i\sin(\theta))e^{-i\theta}$ with respect to θ.

(b) Explain why the result of the previous step means that $f(\theta)$ is constant.

(c) Evaluate f at $\theta = 0$ to find this constant from the previous step.

(d) Rearrange to get Euler's formula.