Assignment 5 - Solutions

Introduction to Linear Algebra

Material from Lectures 8 and 9
Due Thursday, February 9, 2023

8.2 (✠1.14) Find the complete solution to $A\mathbf{x} = \mathbf{b}$, for

$$A = \begin{bmatrix} 3 & 0 & -9 & -3 & 0 \\ 6 & 0 & -21 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 9 \\ -1 \\ 0 \end{bmatrix}.$$

First we find a particular solution. We get these by elimination on the augmented matrix $[A\ \mathbf{b}]$. The first multiplier is $\ell_{21} = 2$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -9 & -3 & 0 & 9 \\ 6 & 0 & -21 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -9 & -3 & 0 & 9 \\ 0 & 0 & -3 & 6 & 2 & -19 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see the pivots already as $3, -3$. Now we clear the -9 above the -3:

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -9 & -3 & 0 & 9 \\ 0 & 0 & -3 & 6 & 2 & -19 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & -21 & -6 & 66 \\ 0 & 0 & -3 & 6 & 2 & -19 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally we multiply by the reciprocals of the pivots:

$$\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & -21 & -6 & 66 \\ 0 & 0 & -3 & 6 & 2 & -19 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -7 & -2 & 22 \\ 0 & 0 & 1 & -2 & -2/3 & 19/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We find a particular solution immediately by using the last column in the pivot rows:

$$\mathbf{p} = \begin{bmatrix} 22 \\ 0 \\ 19/3 \\ 0 \\ 0 \end{bmatrix}.$$

The special solutions, which we know there are 3 (as there are 3 free columns), come from considering $R\mathbf{x} = 0$. The three special solutions will have one 1 in each of the free variable spots, and 0 in the other free variable spots.

$$\mathbf{s}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{s}_2 = \begin{bmatrix} 7 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{s}_3 = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the complete solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{bmatrix} 22 \\ 0 \\ 19/3 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{bmatrix},$$

for any $c_1, c_2, c_3 \in \mathbf{R}$.

8.5 (✠1.15) Consider the vectors $\mathbf{a} = \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b \\ b \\ b \\ b \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, for $a, b \in \mathbf{R}$.

(a) What will be the rank of the following 4×4 matrices:

i. $\mathbf{a}\mathbf{u}^T$ ii. $\mathbf{b}\mathbf{v}^T$ iii. $\mathbf{a}\mathbf{u}^T + \mathbf{b}\mathbf{v}^T$

Your answers should depend on a and b.

(b) Explain why the rank of $\mathbf{x}\mathbf{y}^T$, for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, can never be greater than 1.

(a) We multiply the given vectors as usual:

$$\mathbf{a}\mathbf{u}^T = \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} a & a & a & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $a = 0$, then we have the zero matrix, which has rank 0. But if a is any nonzero real number, then $\mathbf{a}\mathbf{u}^T$ has only one pivot, so the rank is 1. In the second case:

$$\mathbf{b}\mathbf{v}^T = \begin{bmatrix} b \\ b \\ b \\ b \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} b & b & b & b \\ 0 & 0 & 0 & 0 \\ b & b & b & b \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} b & b & b & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As before, if $b = 0$, the rank is 0. If $b \neq 0$, the rank is 1. In the third case:

$$\mathbf{a}\mathbf{u}^T + \mathbf{b}\mathbf{v}^T = \begin{bmatrix} a+b & a+b & a+b & a+b \\ a & a & a & a \\ a+b & a+b & a+b & a+b \\ a & a & a & a \end{bmatrix}.$$

There are five scenarios:

- If $a = b = 0$, then rank is 0.
- If $a = 0$ and $b \neq 0$, then we are in the second case, and rank is 1.
- If $a \neq 0$ and $b = 0$, we are in the first case, and rank is 1.
- If $a \neq 0$ and $b \neq 0$ but $a + b = 0$, then rank is 1, as we have two zero rows and two rows of just a.
- If $a \neq 0$ and $b \neq 0$ and $a+b \neq 0$, then rows 2, 3, 4 can be made zero by subracting row 1 (in the case of row 3) and dividing by $a + b$ and then subtracting (in the case of rows 2,4). Hence rank is 1.

(b) The rank can never be greater than 1 because every row has the same number in every column. That is, any nonzero row can be used to make any other nonzero row into a zero row, so there can never be more than 1 nonzero rows after Gaussian elimination.

9.2 (✠4.02) Prove all the claims of Proposition 9.5, for $z = x + yi, w = a + bi \in \mathbf{C}$:

(a) $\overline{z+w} = \overline{z} + \overline{w}$

(b) $\overline{zw} = \overline{z}\,\overline{w}$

(c) $\overline{\overline{z}} = z$

(d) $z + \overline{z} = 2x$

(e) $z - \overline{z} = 2yi$

(f) $z^{-1} = \overline{z}/|z|^2$ for $z \neq 0$

(g) $|z| = 0$ iff $z = 0$

(h) $|\overline{z}| = |z|$

(i) $|zw| = |z||w|$

(j) $|z + w| \leqslant |z| + |w|$

(a)

$$\begin{aligned}\overline{z+w} &= \overline{(x+yi)+(a+bi)} \\ &= \overline{(x+a)+(y+b)i} \\ &= (x+a)-(y+b)i \\ &= (a-yi)+(a-bi) \\ &= \overline{z}+\overline{w}\end{aligned}$$

(b)

$$\begin{aligned}\overline{zw} &= \overline{(x+yi)(a+bi)} \\ &= \overline{xa+xbi+yai-yb} \\ &= \overline{(xa-yb)+(xb+ya)i} \\ &= (xa-yb)-(xb+ya)i \\ &= xa-yb-xbi-yai \\ &= (x-yi)a-(x-yi)bi \\ &= (x-yi)(a-bi) \\ &= \overline{z}\,\overline{w}\end{aligned}$$

(c)

$$\overline{\overline{z}} = \overline{\overline{x+yi}} = \overline{x-yi} = x+yi = z$$

(d)

$$z+\overline{z} = (x+yi)+(x-yi) = (x+x)+(y-y)i = 2x$$

(e)

$$z-\overline{z} = (x+yi)-(x-yi) = (x-x)+(y+y)i = 2yi$$

(f) Since $zz^{-1} = 1$, we have that

$$z^{-1} = \frac{1}{z} = \frac{1}{x+yi} = \frac{1}{x+yi}\,\frac{x-yi}{x-yi} = \frac{x-yi}{x^2+y^2} = \frac{\overline{z}}{|z|^2}.$$

(g) Suppose that $|z| = 0$. Then

$$0 = |z| = \sqrt{x^2+y^2} \quad\Longrightarrow\quad 0 = x^2+y^2.$$

Since $x^2 \geqslant 0$ and $y^2 \geqslant 0$, but their sum is equal to zero, it must be that $x = y = 0$, so $z = 0$. Conversely, suppose that $z = 0$. Then $|z| = \sqrt{0^2} = 0$.

(h)

$$|\overline{z}| = |\overline{x+yi}| = |x-yi| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |x+yi| = |z|$$

(i)

$$
\begin{aligned}
|zw| &= |(x+yi)(a+bi)| \\
&= |xa + xbi + yai - yb| \\
&= |(xa - yb) + (xb + ya)i| \\
&= \sqrt{(xa-yb)^2 + (xb+ya)^2} \\
&= \sqrt{(xa)^2 - 2xayb + (yb)^2 + (xb)^2 + 2xbya + (ya)^2} \\
&= \sqrt{(xa)^2 + (yb)^2 + (xb)^2 + (ya)^2} \\
&= \sqrt{(x^2+y^2)(a^2+b^2)} \\
&= \sqrt{x^2+y^2}\sqrt{a^2+b^2} \\
&= |z||w|
\end{aligned}
$$

(j) For this question we work backwards, doing invertible operations (adding / subtracting, multipliying / dividing by nonzero numbers):

$$
\begin{aligned}
& & |z+w| &\leqslant |z| + |w| & \\
&\iff & |(x+yi) + (a+bi)| &\leqslant |x+yi| + |a+bi| & \text{(expanding)} \\
&\iff & |(x+a) + (y+b)i| &\leqslant |x+yi| + |a+bi| & \text{(expanding)} \\
&\iff & \sqrt{(x+a)^2 + (y+b)^2} &\leqslant \sqrt{x^2+y^2} + \sqrt{a^2+b^2} & \text{(definition)} \\
&\iff & (x+a)^2 + (y+b)^2 &\leqslant x^2 + y^2 + 2\sqrt{(x^2+y^2)(a^2+b^2)} + a^2 + b^2 & \text{(squaring)} \\
&\iff & x^2 + 2ax + a^2 + y^2 + 2yb + b^2 &\leqslant x^2 + y^2 + 2\sqrt{(x^2+y^2)(a^2+b^2)} + a^2 + b^2 & \text{(expanding)} \\
&\iff & 2ax + 2yb &\leqslant 2\sqrt{(x^2+y^2)(a^2+b^2)} & \text{(cancelling)} \\
&\iff & ax + yb &\leqslant \sqrt{(x^2+y^2)(a^2+b^2)} & \text{(dividing by 2)} \\
&\iff & (ax)^2 + 2axyb + (yb)^2 &\leqslant x^2a^2 + x^2b^2 + y^2a^2 + y^2b^2 & \text{(squaring)} \\
&\iff & 2axyb &\leqslant x^2b^2 + y^2a^2 & \text{(cancelling)} \\
&\iff & 0 &\leqslant x^2b^2 - 2axyb + y^2a^2 & \text{(rearranging)} \\
&\iff & 0 &\leqslant (xb - ya)^2 & \text{(rearranging)}
\end{aligned}
$$

This last line is clearly a true statement, and since all operations were reversible, the first line is also true.

9.3 (✠4.01) This question is about proving Euler's formula $\cos(\theta) + i\sin(\theta) = e^{i\theta}$.

(a) Take the derivative of $f(\theta) = (\cos(\theta) + i\sin(\theta))e^{-i\theta}$ with respect to θ.

(b) Explain why the result of the previous step means that $f(\theta)$ is constant.

(c) Evaluate f at $\theta = 0$ to find this constant from the previous step.

(d) Rearrange to get Euler's formula.

(a) We use the product rule and the chain rule here:

$$\begin{aligned}
\frac{d}{d\theta}\left((\cos(\theta) + i\sin(\theta))e^{-i\theta}\right) &= \left(\frac{d}{d\theta}(\cos(\theta) + i\sin(\theta))\right)e^{-i\theta} + (\cos(\theta) + i\sin(\theta))\left(\frac{d}{d\theta}e^{-i\theta}\right) \\
&= \left(\frac{d}{d\theta}\cos(\theta) + i\frac{d}{d\theta}\sin(\theta)\right)e^{-i\theta} + (\cos(\theta) + i\sin(\theta))\left(-ie^{-i\theta}\right) \\
&= (-\sin(\theta) + i\cos(\theta))\,e^{-i\theta} + (\cos(\theta) + i\sin(\theta))\left(-ie^{-i\theta}\right) \\
&= -\sin(\theta)e^{-i\theta} + i\cos(\theta)e^{-i\theta} - i\cos(\theta)e^{-i\theta} + \sin(\theta)e^{-i\theta} \\
&= 0.
\end{aligned}$$

(b) By the fundamental theorem of calculus, integrals and derivatives are inverses of each other. Taking the derivative of 0 we get a constant C, and this consant must be equal to $f(\theta)$.

(c) Following the prompt, we see

$$f(0) = (\cos(0) + i\sin(0))e^{-i0} = (1 + i \cdot 0) \cdot 1 = 1.$$

Since $f(0) = 1$, and since $f(\theta)$ is constant for all θ, it follows that $f(\theta) = 1$.

(d) Following the prompt, we see

$$1 = (\cos(\theta)+i\sin(\theta))e^{-i\theta} \iff \frac{1}{e^{-i\theta}} = \cos(\theta)+i\sin(\theta) \iff e^{i\theta} = \cos(\theta)+i\sin(\theta),$$

which is Euler's formula.