Assignment 4 - Solutions

Introduction to Linear Algebra

Material from Lectures 6 and 7

Due Thursday, February 2, 2023

6.2 (✠2.02) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three different vectors in a vector space V. Consider the three spans $S_1 = \text{span}(\{\mathbf{u}-\mathbf{v}\})$, $S_2 = \text{span}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\})$ and $S_3 = \text{span}(\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}\})$.

(a) Show that $S_1 \subseteq S_2$.

(b) Show that $S_3 \subseteq S_2$.

(c) For $V = \mathbf{R}^3$, given an example of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ for which $S_2 = S_3$.

(d) For $V = \mathbf{R}^3$, given an example of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ for which all of S_1, S_2, S_3 are different.

(a) To show this, we need to show that every element in S_1 can be expressed an element in S_2. An arbitrary element of S_1 looks like $a(\mathbf{u}-\mathbf{v})$ for some $a \in \mathbf{R}$. An arbitrary element of S_2 looks like $b\mathbf{u} + c\mathbf{v} + d\mathbf{w}$ for some $b, c, d \in \mathbf{R}$. For the choices $b = a$, $c = -a$, and $d = 0$, we get

$$b\mathbf{u} + c\mathbf{v} + d\mathbf{w} = a\mathbf{u} + (-a)\mathbf{v} + 0\mathbf{w} = a(\mathbf{u}-\mathbf{v}),$$

which was the arbitrary element of S_1 that we began with. Therefore $S_1 \subseteq S_2$.

(b) To show this, we need to show that every element in S_3 can be expressed an element in S_2. An arbitrary element of S_3 looks like $a(\mathbf{u}+\mathbf{v}) + b(\mathbf{v}+\mathbf{w})$ for some $a, b \in \mathbf{R}$. An arbitrary element of S_2 looks like $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$ for some $c, d, e \in \mathbf{R}$. For the choice $c = a$, $d = a + b$, and $d = b$, we get

$$c\mathbf{u} + d\mathbf{v} + e\mathbf{w} = a\mathbf{u} + (a+b)\mathbf{v} + b\mathbf{w} = a(\mathbf{u}+\mathbf{v}) + b(\mathbf{v}+\mathbf{w}),$$

which was the arbitrary element of S_3 that we began with. Therefore $S_3 \subseteq S_2$.

(c) There are lots of examples, with the key being that one of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ must be a linear combination of the other two. For example, we could choose

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{u}+\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}+\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Since $\mathbf{u} = \mathbf{w} - \mathbf{v}$ (that is, $\mathbf{u}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$), it follows that $S_2 = \text{span}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}) = \text{span}(\{\mathbf{v}, \mathbf{w}\})$. Since $\mathbf{u}+\mathbf{v} = \mathbf{w}$ (by how the vectors are defined) and $(\mathbf{v}+\mathbf{w}) = \mathbf{v}+\mathbf{w}$ (that is, $\mathbf{v}+\mathbf{w}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$), it follows that $S_3 = \text{span}(\{\mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}\}) = \text{span}(\{\mathbf{v}, \mathbf{w}\})$. Hence $S_2 = S_3$.

(d) There are lots of examples, with the key being that none of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ can be linear combinations of the other two. For example, we could choose

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{u}-\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{u}+\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}+\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The vector $\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$ is not in the span of $\left[\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right]$, so $S_1 \neq S_2$. The vector $\left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]$ is not in the span of $\left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right]$, so $S_3 \neq S_2$. Finally, the vector $\left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right]$ is not in the span of $\left[\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right]$, so $S_1 \neq S_3$.

6.3 (✠2.01) Consider the set X of all functions $f\colon \mathbf{R} \to \mathbf{R}$.

(a) If addition on X is defined as $(f+g)(x) = f(x) + g(x)$ and multiplication is defined as $(cf)(x) = f(cx)$, show that X can not be a vector space.

(b) If multiplication is instead defined as $(cf)(x) = cf(x)$, and addition is instead defined as $(f+g)(x) = f(g(x))$ show that X still can not be a vector space.

Hint: Show X is not a vector space with examples!

(a) This is not a vector space, because scalar multiplication is not distributive over field addition. For example, if $f(x) = x^2 - 1$, $a = 3$, $b = -2$, then

$$\begin{aligned}(a+b)f(x) &= (3 + (-2))f(x) = 1f(x) = f(1x) = x^2 - 1,\\ af(x) + bf(x) &= 3f(x) + (-2)f(x) = f(3x) + f(-2x) = (3x)^2 - 1 + (-2x)^2 - 1\\ &= 9x^2 + 4x^2 - 2 = 13x^2 - 2,\end{aligned}$$

and these are clearly not the same function.

(b) This is not a vector space, because addition is not commutative. For example, if $f(x) = x^2$ and $g(x) = 2x$, then

$$\begin{aligned}f + g &= f(g(x)) = f(2x) = 4x^2,\\ g + f &= g(f(x)) = g(x^2) = 2x^2,\end{aligned}$$

which are clearly not the same function.

7.1 (✠1.12) Consider the matrix $A = \begin{bmatrix} 2 & 9 & 1 & 0 & 0 & 9 \\ 0 & -3 & 0 & 1 & 0 & -3 \\ 8 & -6 & 0 & 0 & 1 & 1 \end{bmatrix}$.

(a) Construct the column space of A as a span of three vectors.

(b) Construct the nullspace of A as a span of vectors.

We take advantage of the fact that there is a 3×3 identity matrix in columns 3-5.

(a) The column space of A is the span of all the columns of A. Columns 1,2,6 can each be constructed as linear combinations of columns 3,4,5, in the sense of

$$\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 8 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 9 \\ -3 \\ -6 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 9 \\ -3 \\ 1 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the column space of A is

$$\mathrm{col}(A) = \mathrm{span}\left(\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ 1 \end{bmatrix} \right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

(b) Consider columns 3,4,5 as our pivot columns. That is, columns 1,2,6 are free columns. The first free column gives us the first vector generating the nullspace, as

$$\begin{bmatrix} 2 & 9 & 1 & 0 & 0 & 9 \\ 0 & -3 & 0 & 1 & 0 & -3 \\ 8 & -6 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ -8 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The second free column gives us the second vector, as

$$\begin{bmatrix} 2 & 9 & 1 & 0 & 0 & 9 \\ 0 & -3 & 0 & 1 & 0 & -3 \\ 8 & -6 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -9 \\ 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

And the third free column gives us the third vector, as

$$\begin{bmatrix} 2 & 9 & 1 & 0 & 0 & 9 \\ 0 & -3 & 0 & 1 & 0 & -3 \\ 8 & -6 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -9 \\ 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

These are all linearly independent, as witnessed by their rows 1,2,6, which are zeros for all but exactly one vector. Hence

$$\mathrm{null}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -9 \\ 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -9 \\ 3 \\ -1 \\ 1 \end{bmatrix} \right).$$

7.6 (✠1.13) Let X be a set of 2×2 matrices defined in the following way:

- $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in X$
- if $M \in X$, then $MM^T \in X$
- if $M, N \in X$, then $aM + bN \in X$, for any $a, b \in \mathbf{R}$

Using scalar multiplication and matrix addition as in $\mathcal{M}_{2\times 2}$, show that X is a vector subspace of $\mathcal{M}_{2\times 2}$.

Hint: Using the given facts, try to construct the four special matrices that generate $\mathcal{M}_{2\times 2}$.

The "four special matrices" mentioned are $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. We construct each of them using the three given rules.

The first and second rule with $M = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ give $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right] \in X$.
The third rule with $a = 1$, $M = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, $b = -1$, $N = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ gives $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right] \in X$.
The second rule with $M = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ gives $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \in X$.
The third rule with $a = 1$, $M = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, $b = -1$, $N = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ gives $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \in X$.
We have now shown one of the four special matrices is in X.

The third rule with $a = 1$, $M = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, $b = -1$, $N = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ gives $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] \in X$.
We have now shown two of the four special matrices are in X.

The third rule with $a = 1$, $M = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, $b = -1$, $N = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ gives $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in X$.
The third rule with $a = 1$, $M = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, $b = -1$, $N = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ gives $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right] \in X$.
The second rule with $M = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right]$ gives $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right] \in X$.
The third rule with $a = 1$, $M = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]$, $b = -1$, $N = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ gives $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \in X$.
We have now shown three of the four special matrices are in X.

The third rule with $a = 1$, $M = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, $b = -1$, $N = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ gives $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in X$.
We have now shown all of the four special matrices are in X.

We now claim that $X = \mathcal{M}_{2\times 2}$. Indeed, any matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathcal{M}_{2\times 2}$ can be constructed using rule 3 three times:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \left(b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \left(c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right).$$

Similarly, X is closed under scalar multiplcation by rule 3 using any number for a and $b = 0$. The existence of the additive inverse and the additive identity follow as X is equal to $\mathcal{M}_{2\times 2}$ as sets. Therefore X is a vector subspace of $\mathcal{M}_{2\times 2}$, in fact it is equal to $\mathcal{M}_{2\times 2}$.