Assignment 3 - Solutions

Introduction to Linear Algebra

Material from Lectures 4 and 5

Due Thursday, January 26, 2023

4.1 (✠1.08) Consider the matrix equation $A\mathbf{x} = \mathbf{b}$, given by $\begin{bmatrix} 3 & -1 & 2 \\ 9 & -3 & 2 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ -3 \end{bmatrix}$. Use Gaussian elimination on the augmented matrix $[A \mid \mathbf{b}]$ to solve for x, y, z.

Following the algorithm for Gausian elimination, notice that $A_{21} = 9$ and $A_{11} = 3$, so we need to multiply the augmented matrix by the row operation that subtracts 3 times the first row from the second row:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 & 5 \\ 9 & -3 & 2 & 5 \\ 1 & -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & -4 & -10 \\ 1 & -1 & -1 & -3 \end{bmatrix}.$$

Although we have a 0 in row 2, column 2, since the goal is to solve for x, y, z, we can continue without swapping rows. The next step is to subtract $\frac{1}{3}$ of the first row from the third row:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & -4 & -10 \\ 1 & -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & -4 & -10 \\ 0 & -\frac{2}{3} & -\frac{5}{3} & -\frac{14}{3} \end{bmatrix}.$$

Clearing "below the diagonal" is done, now we clear "above the diagonal." First we get a 0 in the row 3, column 3 position:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{5}{12} & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & -4 & -10 \\ 0 & -\frac{2}{3} & -\frac{5}{3} & -\frac{14}{3} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & -4 & -10 \\ 0 & -\frac{2}{3} & 0 & -\frac{1}{2} \end{bmatrix}.$$

Next, we get a 0 in the row 1, column 3 position:

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & -4 & -10 \\ 0 & -\frac{2}{3} & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 0 & -4 & -10 \\ 0 & -\frac{2}{3} & 0 & -\frac{1}{2} \end{bmatrix}.$$

Finally, we get a 0 in the row 1, column 2 position:

$$\begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 0 & -4 & -10 \\ 0 & -\frac{2}{3} & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & -4 & -10 \\ 0 & -\frac{2}{3} & 0 & -\frac{1}{2} \end{bmatrix}.$$

This gives us that the solution is $x = \frac{1}{4}$, $z = \frac{10}{4} = \frac{5}{2}$, and $y = \frac{3}{4}$.

4.4 (✠1.09) Using Gauss–Jordan elimination, find the inverse matrix of $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & -4 \\ 2 & 2 & 2 \end{bmatrix}$.

We apply row operations to the block matrix $[A\ I] = \left[\begin{smallmatrix} 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & 0 & -4 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{smallmatrix}\right]$, as below.

swap the first and the second rows to get a first pivot: $\begin{bmatrix} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{bmatrix}$

subtract 2 times the first row from the third row: $\begin{bmatrix} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 10 & 0 & -2 & 1 \end{bmatrix}$

subtract the second row from the third row: $\begin{bmatrix} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 11 & -1 & -2 & 1 \end{bmatrix}$

This finishes Gaussian elmination, so we proceeed with Gauss–Jordan elimination above the diagonal.

subtract $-1/11$ times the third row from the second row: $\begin{bmatrix} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & 0 & 10/11 & -2/11 & 1/11 \\ 0 & 0 & 11 & -1 & -2 & 1 \end{bmatrix}$

subtract $-4/11$ times the third row from the first row: $\begin{bmatrix} 1 & 0 & 0 & -4/11 & 3/11 & 1/11 \\ 0 & 2 & 0 & 10/11 & -2/11 & 1/11 \\ 0 & 0 & 11 & -1 & -2 & 1 \end{bmatrix}$

multiply each row by the inverse of the pivots: $\begin{bmatrix} 1 & 0 & 0 & -4/11 & 3/11 & 1/11 \\ 0 & 1 & 0 & 5/11 & -1/11 & 1/22 \\ 0 & 0 & 1 & -1/11 & -2/11 & 1/11 \end{bmatrix}$

Hence the inverse of A is $A^{-1} = \left[\begin{smallmatrix} -4/11 & 3/11 & 1/11 \\ 5/11 & -1/11 & 1/22 \\ -1/11 & -2/11 & 1/11 \end{smallmatrix}\right]$.

5.2 (✠1.10) Decompose $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ into $PA = LDU$ factorization.

First we clear below the diagonal. Attempting to get a 0 in the $(2,1)$-position, we run into a problem, as the second row has two zeros in front (instead of the desired one):

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

So we undo this step, and swap the second and third rows. This will ensure that L is indeed lower triangular at the end:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}.$$

Now we proceed as usual, first clearing the $(2,1)$-position:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{4} \\ 2 & 1 & 2 \end{bmatrix}$$

Next we clear the $(3,1)$-position, noticing that this clears the $(3,2)$-position as well:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E_2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{4} \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}.$$

So far we have an equation $E_2E_1PA =$ (upper triangular), but we need to get to $PA =$ (lower triangular)(upper triangular). This means moving E_2, then E_1 from the left to the right, in the following way:

$$\begin{aligned}
E_2E_1PA &= \text{(upper triangular)} && \text{(given equation)} \\
E_2^{-1}E_2E_1PA &= E_2^{-1}\text{(upper triangular)} && \text{(multiply by } E_2^{-1} \text{ on both sides)} \\
IE_1PA &= E_2^{-1}\text{(upper triangular)} && \text{(definition of the inverse)} \\
E_1PA &= E_2^{-1}\text{(upper triangular)} && \text{(properties of the identity)} \\
E_1^{-1}E_1PA &= E_1^{-1}E_2^{-1}\text{(upper triangular)} && \text{(multiply by } E_1^{-1} \text{ on both sides)} \\
IPA &= E_1^{-1}E_2^{-1}\text{(upper triangular)} && \text{(definition of the inverse)} \\
PA &= E_1^{-1}E_2^{-1}\text{(upper triangular)} && \text{(properties of the identity)}
\end{aligned}$$

Inverses of elimination matrices are the reverse operations (adding instead of subtracting), hence:

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying these two together in the order $E_1^{-1}E_2^{-1}$ gives the lower triangular matrix L on the right:

$$E_1^{-1}E_2^{-1} = \begin{bmatrix}1 & 0 & 0\\ \frac{1}{2} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ \frac{1}{2} & 1 & 0\\ 1 & 0 & 1\end{bmatrix} = L.$$

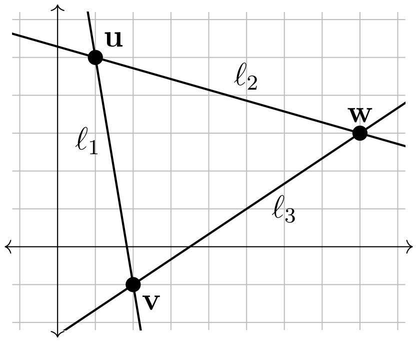
We now have $PA = L$(upper triangular), but the upper triangular matrix does not have 1's on its diagonal. That is, weneed to factor out the leading coefficient in each row:

$$\underbrace{\begin{bmatrix}2 & 1 & 1\\ 0 & \frac{1}{2} & \frac{3}{4}\\ 0 & 0 & 1\end{bmatrix}}_{\text{upper triangular}} = \underbrace{\begin{bmatrix}2 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & \frac{3}{2}\\ 0 & 0 & 1\end{bmatrix}}_{U}.$$

Putting this all together, we finally have the necessary decomposition:

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}2 & 1 & 1\\ 2 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ \frac{1}{2} & 1 & 0\\ 1 & 0 & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}2 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & \frac{3}{2}\\ 0 & 0 & 1\end{bmatrix}}_{U}.$$

5.6 (✠1.06, 1.11) Consider three points $\mathbf{u} = (1,5), \mathbf{v} = (2,-1), \mathbf{w} = (8,3)$ in $\mathbf{R}^2$. Let ℓ_1 be the line through $\mathbf{u}$ and $\mathbf{v}$, ℓ_2 be the line through $\mathbf{u}, \mathbf{w}$, and ℓ_3 be the line through $\mathbf{v}, \mathbf{w}$, as in the diagram below.



(a) Give the matrix equation for which the lines in the diagram above are the row picture.

(b) Without solving this matrix equation, explain why the the equation has no solutions.

(c) Now suppose that $\mathbf{u} = (5,1)$. Give the new matrix equation (the lines ℓ_1, ℓ_2, ℓ_3 are constructed in the same way), and again, without solving it, explain why it has infinitely many solutions.

(a) For the row picture, we need the equations of the lines. Since we have two points for each line, we have the slope, and so the equation can be found by the point-slope formula

$$\frac{x-x_1}{y-y_1} = \frac{x_2-x_1}{y_2-y_1} \qquad \text{or} \qquad x - x_1 = m(y-y_1),$$

where $m = \frac{x_2-x_1}{y_2-y_1}$ is the slope, given the two points (x_1,y_1) and (x_2,y_2) on the line. For ℓ_1, we have

$$\frac{x-1}{y-5} = \frac{2-1}{(-1)-5} \implies x-1 = -\frac{1}{6}y + \frac{5}{6} \implies x + \frac{1}{6}y = \frac{11}{6}.$$

For ℓ_2, we have

$$\frac{x-1}{y-5} = \frac{8-1}{3-5} \implies x-1 = -\frac{7}{2}y + \frac{35}{2} \implies x + \frac{7}{2}y = \frac{37}{2}$$

For ℓ_3, we have

$$\frac{x-2}{y-(-1)} = \frac{8-2}{3-(-1)} \implies x-2 = \frac{3}{2}y + \frac{3}{2} \implies x - \frac{3}{2}y = \frac{7}{2}.$$

Putting this all together, the desired matrix equation is

$$\begin{bmatrix} 1 & \frac{1}{6} \\ 1 & \frac{7}{2} \\ 1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{11}{6} \\ \frac{37}{2} \\ \frac{7}{2} \end{bmatrix}.$$

(b) The matrix equation has no solutions because the three lines it represents (in the row picture) do not all intersect in a single point. That is, no point in $\mathbf{R}^2$ is on all three lines simultaneously, so no solution to the matrix equation exists.

(c) For this part, we need to reconstruct ℓ_1 and $\ell 2$. For the new ℓ_1, we have

$$\frac{x-5}{y-1} = \frac{2-5}{(-1)-1} \quad \Longrightarrow \quad x-5 = \frac{3}{2}y - \frac{3}{2} \quad \Longrightarrow \quad x - \frac{3}{2}y = \frac{7}{2}.$$

For the new ℓ_2, we have

$$\frac{x-5}{y-1} = \frac{8-5}{3-1} \quad \Longrightarrow \quad x-5 = \frac{3}{2}y - \frac{3}{2} \quad \Longrightarrow \quad x - \frac{3}{2}y = \frac{7}{2}.$$

The line ℓ_3 stays the same. The new matrix equation is

$$\begin{bmatrix} 1 & -\frac{3}{2} \\ 1 & -\frac{3}{2} \\ 1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ \frac{7}{2} \\ \frac{7}{2} \end{bmatrix}.$$

This matrix equation has infinitely many solutions, because the three lines ℓ_1, ℓ_2, ℓ_3 are now coincident, that is, they lie on top of each other. This follows from the observation that the new coordinates $(5, 1)$ of $\mathbf{u}$ are on ℓ_3, as in the picture below.