Assignment 12 - Solutions

Introduction to Linear Algebra

Material from Lectures 22 and 23

Due Thursday, April 6, 2023

22.2 (✠3.08) Let A, B, C be any 3×3 matrices, with C diagonalizable.

(a) Show that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$.

(b) Use that above to show that $\operatorname{trace}(C)$ is the sum of the three eigenvalues of C. *Hint: Split up the diagonalization of C into two matrices.*

(c) Suppose that the eigenvalues of C are $1, \frac{1}{2}, \frac{1}{3}$. Show why the limit $\lim_{n\to\infty} C^n$ exists, and why it has rank 1.

(a) This follows by computing the diagonal entries of AB and of BA. By the formula for entries in a product of matrices:

$$(AB)_{ii} = \sum_{j=1}^{3} A_{ij}B_{ji} = A_{i1}B_{1i} + A_{i2}B_{2i} + A_{i3}B_{3i},$$
$$(BA)_{ii} = \sum_{j=1}^{3} B_{ij}A_{ji} = B_{i1}A_{1i} + B_{i2}A_{2i} + B_{i3}A_{3i}.$$

Summing up for $i = 1, 2, 3$, for the trace, we find that

$$\begin{aligned}
\operatorname{trace}(AB) &= (AB)_{11} + (AB)_{22} + (AB)_{33} \\
&= A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} + A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} + A_{31}B_{13} + A_{32}B_{23} + A_{33}B_{33}, \\
\operatorname{trace}(BA) &= (BA)_{11} + (BA)_{22} + (BA)_{33} \\
&= B_{11}A_{11} + B_{12}A_{21} + B_{13}A_{31} + B_{21}A_{12} + B_{22}A_{22} + B_{23}A_{32} + B_{31}A_{13} + B_{32}A_{23} + B_{33}A_{33},
\end{aligned}$$

which are the same.

(b) Since C is diagonalizable, there exists an invertible matrix X of the eigenvectors of C as columns, and a diagonal matrix Λ of the eigenvalues of C on its diagonal, with $C = X\Lambda X^{-1}$. Using the previous task with $A = X$ and $B = \Lambda X^{-1}$, we get that

$$\begin{aligned}
\operatorname{trace}(C) &= \operatorname{trace}(X\Lambda X^{-1}) && \text{(since } C \text{ is diagonalizable)} \\
&= \operatorname{trace}((A)(\Lambda X^{-1})) \\
&= \operatorname{trace}((\Lambda X^{-1})(X)) && \text{(by part (a) above)} \\
&= \operatorname{trace}(\Lambda X^{-1} X) \\
&= \operatorname{trace}(\Lambda I) && \text{(definition of the inverse)} \\
&= \operatorname{trace}(\Lambda) \\
&= \Lambda_{11} + \Lambda_{22} + \Lambda_{33}. && \text{(since } \Lambda \text{ is diagonal)}
\end{aligned}$$

This is the sum of the eigenvalues of C, since the eigenvalues of C are on the diagonal of Λ.

(c) If the eigenvalues of C are $1, \frac{1}{2}, \frac{1}{3}$, then the eigenvalues of C^n are $1^n, \frac{1}{2^n}, \frac{1}{3^n}$, as

$$\begin{aligned} C^2 &= (X\Lambda X^{-1})(X\Lambda X^{-1}) = X\Lambda I\Lambda X^{-1} = X\Lambda^2 X^{-1} \\ C^3 &= (X\Lambda^2 X^{-1})(X\Lambda X^{-1}) = X\Lambda^2 I\Lambda X^{-1} = X\Lambda^3 X^{-1} \\ &\vdots \\ C^n &= X\Lambda^n X^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} \lim_{n\to\infty} C^n &= \lim_{n\to\infty} \left(X\Lambda^n X^{-1}\right) \\ &= \lim_{n\to\infty} \left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} 1^n & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{3^n} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \right) \\ &= \lim_{n\to\infty} \left(\begin{bmatrix} x_{11} & \frac{x_{12}}{2^n} & \frac{x_{13}}{3^n} \\ x_{21} & \frac{x_{22}}{2^n} & \frac{x_{23}}{3^n} \\ x_{31} & \frac{x_{32}}{2^n} & \frac{x_{33}}{3^n} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \right) \\ &= \lim_{n\to\infty} \left(\begin{bmatrix} x_{11}y_{11}+\frac{1}{2^n}\cdot x_{12}y_{21}+\frac{1}{3^n}\cdot x_{13}y_{31} & x_{11}y_{12}+\frac{1}{2^n}\cdot x_{12}y_{22}+\frac{1}{3^n}\cdot x_{13}y_{32} & x_{11}y_{13}+\frac{1}{2^n}\cdot x_{12}y_{23}+\frac{1}{3^n}\cdot x_{13}y_{33} \\ x_{21}y_{11}+\frac{1}{2^n}\cdot x_{22}y_{21}+\frac{1}{3^n}\cdot x_{23}y_{31} & x_{21}y_{12}+\frac{1}{2^n}\cdot x_{22}y_{22}+\frac{1}{3^n}\cdot x_{23}y_{32} & x_{21}y_{13}+\frac{1}{2^n}\cdot x_{22}y_{23}+\frac{1}{3^n}\cdot x_{23}y_{33} \\ x_{31}y_{11}+\frac{1}{2^n}\cdot x_{32}y_{21}+\frac{1}{3^n}\cdot x_{33}y_{31} & x_{31}y_{12}+\frac{1}{2^n}\cdot x_{32}y_{22}+\frac{1}{3^n}\cdot x_{33}y_{32} & x_{31}y_{13}+\frac{1}{2^n}\cdot x_{32}y_{23}+\frac{1}{3^n}\cdot x_{33}y_{33} \end{bmatrix} \right) \\ &= \begin{bmatrix} x_{11}y_{11} & x_{11}y_{12} & x_{11}y_{13} \\ x_{21}y_{11} & x_{21}y_{12} & x_{21}y_{13} \\ x_{31}y_{11} & x_{31}y_{12} & x_{31}y_{13} \end{bmatrix} \\ &= \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \end{bmatrix}. \end{aligned}$$

The limit exists, and as it is a (outer) product of two vectors, it must have rank at most 1. Moreover, we know it has rank exactly one, because to have rank zero either the column or row vector must be all zeros - but this is not possible, as then the matrices X, X^{-1} would not be invertible.

22.3 (✠3.09) Decompose both matrices below in their $X\Lambda X^{-1}$-decomposition, where Λ is a diagonal matrix with the eigenvalues, and X is the matrix with columns as eigenvectors.

$$A = \begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

For the matrix A, note that the rows are multiples of each other, so $\lambda_1 = 0$. Since there are 2 eigenvalues (as it is a 2×2 matrix), and the sum of the eigenvalues is the trace, it follows that $\lambda_1 + \lambda_2 = 2 + 5 = 7$, so $\lambda_2 = 7$. The eigenvalues may be also found by finding the roots of the characteristic polynomial $\chi(t) = \det(A - tI)$.

For the eigenvectors, we eliminate the augmented matrices

$$\begin{bmatrix} 2 & 2 & 0 \\ 5 & 5 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -5 & 2 & 0 \\ 5 & -2 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -2/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So the eigenvectors are $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ for $\lambda_1 = 0$ and $\left[\begin{smallmatrix} 2/5 \\ 1 \end{smallmatrix}\right]$ for $\lambda_2 = 7$, giving the decomposition

$$\begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2/5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} -1 & 2/5 \\ 1 & 1 \end{bmatrix}^{-1} \quad \text{where} \quad \begin{bmatrix} -1 & 2/5 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{-5}{7} \begin{bmatrix} 1 & -2/5 \\ -1 & -1 \end{bmatrix}.$$

For the matrix B, the eigenvalues are on the diagonal, but the eigenvectors are not so immediate. For $\lambda_1 = 1$ we have $\mathbf{e}_1$, but for $\lambda_2 = 4$ we need

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4x \\ 4y \\ 4z \end{bmatrix} \implies \begin{aligned} z &= 0 \\ 4y &= 4y \\ -3x &= -2y \end{aligned} \implies \mathbf{v}_2 = \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}.$$

Similarly for $\lambda_3 = 6$, we need

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6x \\ 6y \\ 6z \end{bmatrix} \implies \begin{aligned} 6z &= 6z \\ -2y &= -5z \\ -5x &= -2y - 3z \end{aligned} \implies \mathbf{v}_3 = \begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix}.$$

Hence the decomposition is

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2/3 & 8/5 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2/3 & 8/5 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{bmatrix}^{-1},$$

where

$$\begin{bmatrix} 1 & 2/3 & 8/5 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2/3 & 1/15 \\ 0 & 1 & -5/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

23.1 (✠3.10) The three vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ are linearly independent.

(a) Construct a matrix A with eigensystem $\{(\mathbf{u}, 2), (\mathbf{v}, -1), (\mathbf{w}, 3)\}$.

(b) Give examples of two matrices B, C that are similar to A.

(a) The 3×3 matrix is unknown, so we give its entries names: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

The three eigenpairs give three equations:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}.$$

This gives a system of 9 equations in 9 variables, given as a new matrix equation

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \\ 0 \\ -1 \\ 3 \\ 3 \\ -3 \end{bmatrix}.$$

Although this is a very large matrix, a computer helps us solve it and gives the matrix A as

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} -6 & 4 & -5 \\ -2 & 3 & -2 \\ 8 & -4 & 7 \end{bmatrix}.$$

(b) The matrices $B = XAX^{-1}$ and $C = YAY^{-1}$, for some invertible 3×3 matrices X, Y, will be similar to A. We choose some easy matrices X, Y for which we know the inverses:

$$X = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}, \quad B = XAX^{-1} = \frac{1}{6} \begin{bmatrix} -36 & 16 & -15 \\ -18 & 18 & -9 \\ 96 & -32 & 42 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}, \quad Y^{-1} = \begin{bmatrix} 0 & 0 & 1/4 \\ 0 & 1/3 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}, \quad C = YAY^{-1} = \frac{1}{6} \begin{bmatrix} 42 & -16 & 24 \\ -18 & 18 & -9 \\ -60 & 32 & -36 \end{bmatrix}.$$

Many other similar matrices exist.

23.6 (✠3.08, 3.12) Consider the symmetric matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$.

(a) Find the trace and determinant of A. Do not use a calculator, show your work.

(b) Diagonalize A as $Q\Lambda Q^T$.

(c) Express A as a sum of rank one matrices using the part above.

(a) The trace is the sum of the diagonal values, so $\operatorname{trace}(A) = 0 + 0 + 2 = 2$.

The combinatorial definition of the determinant is a sum of 6 terms, with each term having three entries of A, with exactly one entry in one row and one column. This means that every entry that has the nonzero $(1,3)$-entry will be zero, as it needs either the zero $(2,1)$- or $(2,2)$-entries. Every other term is zero, as the $(1,1)$- and $(1,2)$-entries are zero. Hence $\det(A) = 0$.

(b) To diagonalize, we need to find the eigenvalues, which are the roots of the characteristic polynomial. We find that

$$\chi(t) = \det(A - tI) = \begin{vmatrix} -t & 0 & -2 \\ 0 & -t & 1 \\ -2 & 1 & 2-t \end{vmatrix} = -t^3 + 2t^2 + 5t,$$

and $\chi(t) = 0$ implies that $t = 0$ or $t = 1+\sqrt{6}$ or $t = 1-\sqrt{6}$. To find the eigenvalues, we eliminate the augmented matrices

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 2 & 0 \end{array}\right] \xrightarrow{RR} \left[\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \mathbf{v}_1 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix},$$

$$\left[\begin{array}{ccc|c} -1-\sqrt{6} & 0 & -2 & 0 \\ 0 & -1-\sqrt{6} & 1 & 0 \\ -2 & 1 & 2-1-\sqrt{6} & 0 \end{array}\right] \xrightarrow{RR} \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{1+\sqrt{6}} & 0 \\ 0 & 1 & \frac{-1}{1+\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \mathbf{v}_2 = \begin{bmatrix} \frac{-2}{1+\sqrt{6}} \\ \frac{1}{1+\sqrt{6}} \\ 1 \end{bmatrix},$$

$$\left[\begin{array}{ccc|c} -1+\sqrt{6} & 0 & -2 & 0 \\ 0 & -1+\sqrt{6} & 1 & 0 \\ -2 & 1 & 2-1+\sqrt{6} & 0 \end{array}\right] \xrightarrow{RR} \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{1-\sqrt{6}} & 0 \\ 0 & 1 & \frac{-1}{1-\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \mathbf{v}_3 = \begin{bmatrix} \frac{-2}{1-\sqrt{6}} \\ \frac{1}{1-\sqrt{6}} \\ 1 \end{bmatrix}.$$

We normalize them as $\mathbf{q}_1 = \begin{bmatrix} 0.447 \\ 0.894 \\ 0 \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} -0.487 \\ 0.243 \\ 0.839 \end{bmatrix}$, $\mathbf{q}_3 = \begin{bmatrix} 0.751 \\ -0.375 \\ 0.544 \end{bmatrix}$, so

$$\underbrace{\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0.447 & -0.487 & 0.751 \\ 0.894 & 0.243 & -0.375 \\ 0 & 0.839 & 0.544 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+\sqrt{6} & 0 \\ 0 & 0 & 1-\sqrt{6} \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 0.447 & 0.894 & 0 \\ -0.487 & 0.243 & 0.839 \\ 0.751 & -0.375 & 0.544 \end{bmatrix}}_{Q^T}.$$

(c) Using the above, we have $A = 0\mathbf{q}_1\mathbf{q}_1^T + (1+\sqrt{6})\mathbf{q}_2\mathbf{q}_2^T + (1-\sqrt{6})\mathbf{q}_3\mathbf{q}_3^T$.