Assignment 11

Introduction to Linear Algebra

Material from Lectures 20 and 21
Due Thursday, March 30, 2023

20.1 (✠3.03)Let $a, b, c, d \in \mathbf{R}$. Using elementary matrices (permutation, elimination, diagonal) to bring these matrices to triangular form, compute their determinants.

$$A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} a & b & a \\ a & c & a \\ a & d & a \end{bmatrix} \qquad C = \begin{bmatrix} a & b & c \\ b & 0 & b \\ c & b & a \end{bmatrix}$$

20.3 (✠3.04) Let A be an $n \times n$ matrix, for some $n \in \mathbf{N}$.

(a) Explain why $\det(kA) = k^n \det(A)$, for any real number k.

(b) If A is skew-symmetric, explain why randomly choosing n in the range $[1, 100]$ means $\det(A) = 0$ exactly half of the time.

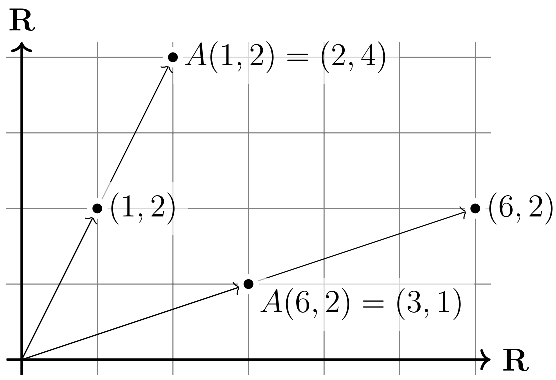
(c) Suppose that A is a projection matrix, projecting from $\mathbf{R}^n$ to an $(n-1)$-dimensional subspace of $\mathbf{R}^n$. Explain why $\det(A) = 0$.

21.5 (✠3.05) Let λ, μ be real numbers, and $\mathbf{u} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \mathbf{v} = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \in \mathbf{R}^2$ be two vectors.

(a) Construct a 2×2 matrix with eigenpairs $(\mathbf{u}, \lambda)$ and $(\mathbf{v}, \mu)$.

(b) What assumptions did you make in the first part to reach a conclusion?

21.6 (✠3.06, 3.07) Let $A \colon \mathbf{R}^2 \to \mathbf{R}^2$ be the 2×2 matrix for which $A\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix} 6 \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$. This is described in the picture below.



(a) What is the eigensystem of A?

(b) Express $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ as linear combinations of the eigenvectors of A.

(c) Compute $A\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Use this to construct the matrix of A.

(d) Using eigenvalues, explain why A is invertible.