Assignment 11 - Solutions

Introduction to Linear Algebra

Material from Lectures 20 and 21

Due Thursday, March 30, 2023

20.1 (✠3.03)Let $a, b, c, d \in \mathbf{R}$. Using elementary matrices (permutation, elimination, diagonal) to bring these matrices to triangular form, compute their determinants.

$$A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} a & b & a \\ a & c & a \\ a & d & a \end{bmatrix} \qquad C = \begin{bmatrix} a & b & c \\ b & 0 & b \\ c & b & a \end{bmatrix}$$

For the matrix A, we swap the 1st and 4th rows, then the 2nd and 4th rows, then the 3rd and 4th rows using permutation matrices:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{determinant } -1} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\text{determinant } -1} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\text{determinant } -1} \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix}}_{\text{determinant } dabc}$$

Hence $\det(A) = -abcd$. For the matrix B, we have to do row reduction with elimination matrices. The last step assumes $c - b \neq 0$. If $c - b = 0$, then the first row reduction operation gives a row of zeros, and so the determinant is still 0.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{d-b}{c-b} & 1 \end{bmatrix}}_{\text{determinant } 1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{\text{determinant } 1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{determinant } 1} \begin{bmatrix} a & b & a \\ a & c & a \\ a & d & a \end{bmatrix} = \underbrace{\begin{bmatrix} a & b & a \\ 0 & c-b & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{determinant } 0}$$

Hence $\det(B) = 0$. For the matrix C, we again use elimination matrices:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{b-\frac{bc}{a}}{-b^2/a} & 1 \end{bmatrix}}_{\text{determinant } 1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{c}{a} & 0 & 1 \end{bmatrix}}_{\text{determinant } 1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{b}{a} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{determinant } 1} \begin{bmatrix} a & b & c \\ b & 0 & b \\ c & b & a \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & -\frac{b^2}{a} & b - \frac{bc}{a} \\ 0 & 0 & a - \frac{c^2}{a} - \frac{(b-\frac{bc}{a})^2}{-b^2/a} \end{bmatrix}$$

To compute the determinant, we multiply the elements on the diagonal:

$$\begin{aligned} \det(C) = a \cdot \left(-\frac{b^2}{a}\right) \cdot \left(a - \frac{c^2}{a} - \frac{(b - \frac{bc}{a})^2}{-b^2/a}\right) &= (-b^2)\left(a - \frac{c^2}{a} - \frac{b^2 - 2b^2c/a + b^2c^2/a^2}{-b^2/a}\right) \\ &= (-b^2)\left(a - \frac{c^2}{a} + \frac{a^2b^2 - 2ab^2c + b^2c^2}{ab^2}\right) \\ &= -b^2 \cdot \frac{a^2b^2 - b^2c^2 + a^2b^2 - 2ab^2c + b^2c^2}{ab^2} \\ &= \frac{-2a^2b^2 + 2ab^2c}{a} \\ &= -2ab^2 + 2b^2c. \end{aligned}$$

20.3 (✠3.04) Let A be an $n \times n$ matrix, for some $n \in \mathbf{N}$.

(a) Explain why $\det(kA) = k^n \det(A)$, for any real number k.

(b) If A is skew-symmetric, explain why randomly choosing n in the range $[1, 100]$ means $\det(A) = 0$ exactly half of the time.

(c) Suppose that A is a projection matrix, projecting from $\mathbf{R}^n$ to an $(n-1)$-dimensional subspace of $\mathbf{R}^n$. Explain why $\det(A) = 0$.

(a) Multiplication by a number is the same as multiplying by the identity matrix with k all along the diagonal. Multiplicativity of the determinant gives the rest:

$$\det(kA) = \det\left(\begin{bmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{bmatrix} A\right) = \det\left(\begin{bmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{bmatrix}\right) \det(A) = k^n \det(A).$$

(b) Since A is skew-symmetric, $A^T = -A$. By properties of the determinant, $\det(A^T) = \det(A)$, and $\det(-A) = (-1)^n \det(A)$ by part (a) above. So for a skew-symmetric matrix we must have $\det(A) = (-1)^n \det(A)$. When n is even, this statement is just $\det(A) = \det(A)$, which doesn't give any information. When n is odd, this statement is $\det(A) = -\det(A)$, which only is true for 0. Hence when n is odd, and exactly half of the integers in the range $[0, 100]$ are odd, $\det(A) = 0$.

(c) Since A is a pojection matrix, by Question 13.1 (from Homework 7), we know that $A^2 = A$. By the multiplicative property of the determinant, $\det(A^2) = \det(A \cdot A) = \det(A)\det(A) = \det(A)^2$. Then

$$\det(A)^2 = \det(A) \quad \Longrightarrow \quad \begin{cases} \det(A) = 1 & \text{if } \det(A) \neq 0 \\ \det(A) = 0 & \text{if } \det(A) = 0. \end{cases}$$

If $\det(A) = 1$, then the n-dimensional volume enclosed by $A\mathbf{v}_1, \ldots, A\mathbf{v}_{2^n}$ is the same as the n-dimensional volume encolused by $\mathbf{v}_1, \ldots, \mathbf{v}_{2^n}$, the corners of the unit n-cube. The statement says we are projecting to an $(n-1)$-dimensional subspace, which has no n-dimensional volume, so $\det(A) \neq 1$. Hence $\det(A) = 0$.

This statement may alternatively be shown by explaining why $\dim(\text{col}(A)) = n - 1$, and so by the rank-nullity theorem $\dim(\text{null}(A)) = 1$. That means one of the columns of A is a linear combination of the other ones, oe equivalently, one of the rows of A^T is a linear combination of the other ones. Row reduction on A^T will give zero row, and since only the sign of the determinant is affected by row reduction, $\det(A^T) = 0$, and so $\det(A) = 0$.

21.5 (✠3.05) Let λ, μ be real numbers, and $\mathbf{u} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \mathbf{v} = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \in \mathbf{R}^2$ be two vectors.

(a) Construct a 2×2 matrix with eigenpairs $(\mathbf{u}, \lambda)$ and $(\mathbf{v}, \mu)$.

(b) What assumptions did you make in the first part to reach a conclusion?

(a) We approach this from the other side. Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be the answer to this question, which will then satisfy

$$A\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \lambda\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \iff \left[\begin{smallmatrix} ax+by \\ cx+dy \end{smallmatrix}\right] = \left[\begin{smallmatrix} \lambda x \\ \lambda y \end{smallmatrix}\right] \quad \text{and} \quad A\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] = \mu\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \iff \left[\begin{smallmatrix} az+bw \\ cz+dw \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mu z \\ \mu w \end{smallmatrix}\right].$$

We could do back substitution, or we could write this as a matrix equation:

$$\begin{bmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \\ z & w & 0 & 0 \\ 0 & 0 & z & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \mu z \\ \mu w \end{bmatrix}.$$

Row reducing the augmented matrix we find solutions to a, b, c, d in the last column:

$$\begin{bmatrix} x & y & 0 & 0 & \lambda x \\ 0 & 0 & x & y & \lambda y \\ z & w & 0 & 0 & \mu z \\ 0 & 0 & z & w & \mu w \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{\mu yz - \lambda xw}{yz - xw} \\ 0 & 1 & 0 & 0 & \frac{\mu xz - \lambda xz}{xw - yz} \\ 0 & 0 & 1 & 0 & \frac{\mu yw - \lambda yw}{yz - xw} \\ 0 & 0 & 0 & 1 & \frac{\mu xw - \lambda yz}{xw - yz} \end{bmatrix}.$$

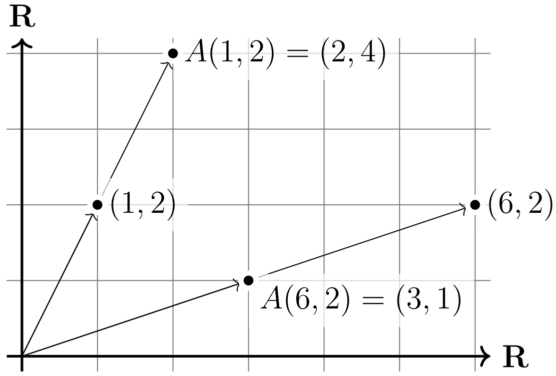
Hence the matrix

$$\begin{bmatrix} \dfrac{\mu yz - \lambda xw}{yz - xw} & \dfrac{\mu xz - \lambda xz}{xw - yz} \\ \dfrac{\mu yw - \lambda yw}{yz - xw} & \dfrac{\mu xw - \lambda yz}{xw - yz} \end{bmatrix}$$

will have eigenvector $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ with eigenvalue λ, and eigenvector $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ with eigenvector μ.

(b) The assumptions made were that the denominators cannot be zero, so that we can divide by them. That is, we assumed $xw - yz \neq 0$, or that $xw \neq yz$.

21.6 (✠3.06, 3.07) Let $A\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the 2×2 matrix for which $A\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix}6\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$. This is described in the picture below.



(a) What is the eigensystem of A?

(b) Express $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ as linear combinations of the eigenvectors of A.

(c) Compute $A\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Use this to construct the matrix of A.

(d) Using eigenvalues, explain why A is invertible.

(a) The eigensystem of A is given by the statement, as two different (linearly independent) eigenvectors with different eigenvalue are given. The eigensystem is $\left\{\left(\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right], 2\right), \left(\left[\begin{smallmatrix}6\\2\end{smallmatrix}\right], \frac{1}{2}\right)\right\}$.

(b) We construct $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ from the given vectors, by clearing the rows with zeros:

$$\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}6\\2\end{bmatrix} = \begin{bmatrix}-5\\0\end{bmatrix} \implies \begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{5}\begin{bmatrix}6\\2\end{bmatrix} - \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix},$$

$$\begin{bmatrix}6\\2\end{bmatrix} - 6\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\-10\end{bmatrix} \implies \begin{bmatrix}0\\1\end{bmatrix} = \frac{6}{10}\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{10}\begin{bmatrix}6\\2\end{bmatrix}.$$

(c) Using the results from part (b), we get what we are asked:

$$A\begin{bmatrix}1\\0\end{bmatrix} = A\left(\frac{1}{5}\begin{bmatrix}6\\2\end{bmatrix} - \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}\right) = \frac{1}{5}A\begin{bmatrix}6\\2\end{bmatrix} - \frac{1}{5}A\begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}3\\1\end{bmatrix} - \frac{1}{5}\begin{bmatrix}2\\4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1\\-3\end{bmatrix},$$

$$A\begin{bmatrix}0\\1\end{bmatrix} = A\left(\frac{6}{10}\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{10}\begin{bmatrix}6\\2\end{bmatrix}\right) = \frac{6}{10}A\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{10}A\begin{bmatrix}6\\2\end{bmatrix} = \frac{6}{10}\begin{bmatrix}2\\4\end{bmatrix} - \frac{1}{10}\begin{bmatrix}3\\1\end{bmatrix} = \frac{1}{10}\begin{bmatrix}9\\23\end{bmatrix}.$$

To get A, we evaluate what it does on an arbitrary vector $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$. First note that

$$A\begin{bmatrix}x\\y\end{bmatrix} = A\left(\begin{bmatrix}x\\0\end{bmatrix} + \begin{bmatrix}0\\y\end{bmatrix}\right) = A\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right) = xA\begin{bmatrix}1\\0\end{bmatrix} + yA\begin{bmatrix}0\\1\end{bmatrix},$$

and now we apply A to these vectors to get

$$x\frac{1}{5}\begin{bmatrix}1\\-3\end{bmatrix} + y\frac{1}{10}\begin{bmatrix}9\\23\end{bmatrix} = \frac{1}{10}\left(x\begin{bmatrix}2\\-6\end{bmatrix} + y\begin{bmatrix}9\\23\end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix}2 & 9\\-6 & 23\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.$$

Hence $A = \frac{1}{10}\left[\begin{smallmatrix}2 & 9\\-6 & 23\end{smallmatrix}\right] = \left[\begin{smallmatrix}1/5 & 9/10\\-3/5 & 23/10\end{smallmatrix}\right]$.

(d) The product of the eigenvalues is the determinant of the matrix. Since $1 \cdot \frac{1}{2} = \frac{1}{2}$, the determinant is $\frac{1}{2}$. By Proposition 20.4 in the lecture notes, A is invertible iff $\det(A) \neq 0$. Hence A is invertible.