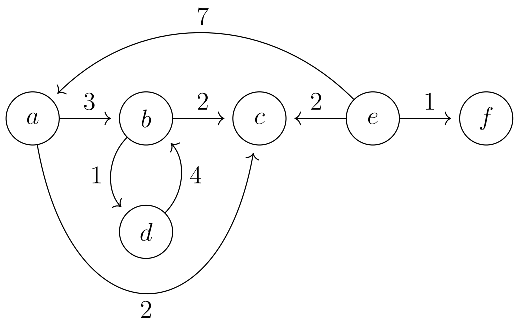
Assignment 10 - Solutions

Introduction to Linear Algebra

Material from Lectures 18 and 19

Due Thursday, March 23, 2023

18.1 (✠4.03, 4.04) Consider the following directed graph:

$G =$ 

(a) Compute the adjacency A, incidence N, Laplacian L, and transition probability T matrices for G. Use the weighted definition for T.

(b) Row reduce N and give the resulting spanning tree of G.

(c) Using a computer make an educated guess as to what $\lim_{n\to\infty} T^n$ could be.

(d) Let $\hat{T}$ be the same as T, but with the (c,c) and (f,f) entries 1 (instead of 0) on the diagonal. Using a computer make an educated guess as to what $\lim_{n\to\infty} \hat{T}^n$ could be.

The last two questions address the *network flow*, where matrix multiplication represents movement along the edges, and the weights represent the probability of moving along a given edge (relative to all the weights outgoing from the tail node).

(a) The graph G has 6 vertices and 8 edges. The adjacency matrix is 6×6:

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

The incidence matrix is 8×6:

$$N = \begin{array}{c} \\ (a,b) \\ (a,c) \\ (b,d) \\ (b,c) \\ (d,b) \\ (e,a) \\ (e,c) \\ (e,f) \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

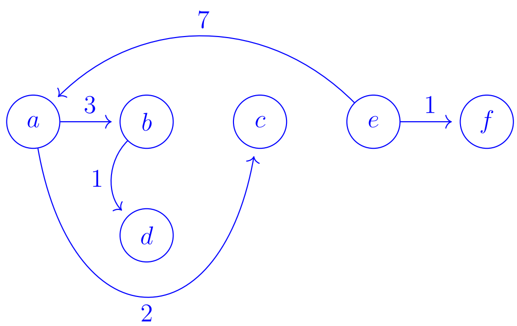
The Laplacian and transition probability matrices are 6×6:

$$L = N^T N = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 4 & -1 & -2 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & -2 & 0 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \qquad T = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & \frac{3}{5} & \frac{2}{5} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{7}{10} & 0 & \frac{2}{10} & 0 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

(b) Row reducing removes the three edges $(b, c), (d, b), (e, c)$:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{G.E.} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

The resulting spanning tree is given below:

$T =$ 

Other spanning trees are possible if different row reduction stapes are taken. All spanning trees should have 5 edges.

(c) Using a computer, we compute the 10th and 100th powers of T:

$$T^{10} = \begin{bmatrix} 0 & 0 & 0.00493 & 0.00246 & 0 & 0 \\ 0 & 0.00411 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.00823 & 0.00411 & 0 & 0 \\ 0 & 0.00518 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad T^{100} = \begin{bmatrix} 0 & 0 & 1.67 \cdot 10^{-24} & 8.36 \cdot 10^{-25} & 0 & 0 \\ 0 & 1.39 \cdot 10^{-24} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.79 \cdot 10^{-24} & 1.39 \cdot 10^{-24} & 0 & 0 \\ 0 & 1.76 \cdot 10^{-24} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since these numbers are so small, we make an educated guess that $\lim_{n \to \infty} T^n = 0$.

(d) With a similar method, we make an educated guess that $\lim_{n \to \infty} \hat{T}^n = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

19.6 (✠3.01) Use the combinatorial definition of the determinant for this question. Recall from Example 19.15 that every term in the combinatorial definition uses exactly one entry from each row and one entry from each column of the matrix.

(a) Compute the determinant of

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

(b) The $n \times n$ identity matrix has $n^2 - n$ zeroes and exactly one nonzero term in the combinatorial definition. What is the smallest number of zeroes an $n \times n$ matrix can have so that the combinatorial definition has only one nonzero term?

(a) Using the hint and the observation that row 3 has one non-zero entry, we know that every nonzero term in the combinatorial definition of the determinant must have this one entry. This means no other row can have its third entry in that nonzero term, so we have more restrictions:

$$\underset{\text{one nonzero choice from row 3}}{\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}} \to \underset{\text{one nonzero choice from row 5}}{\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ \mathbf{1} & 0 & 1 & 0 & 0 \end{bmatrix}} \to \underset{\text{one nonzero choice from row 1}}{\begin{bmatrix} 1 & 0 & 1 & 0 & \mathbf{1} \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ \mathbf{1} & 0 & 1 & 0 & 0 \end{bmatrix}} \to \underset{\text{one nonzero choice from row 4}}{\begin{bmatrix} 1 & 0 & 1 & 0 & \mathbf{1} \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 1 & \mathbf{1} & 1 & 0 & 1 \\ \mathbf{1} & 0 & 1 & 0 & 0 \end{bmatrix}} \to \underset{\text{one nonzero choice from row 2}}{\begin{bmatrix} 1 & 0 & 1 & 0 & \mathbf{1} \\ 1 & 1 & 1 & \mathbf{1} & 1 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 1 & \mathbf{1} & 1 & 0 & 1 \\ \mathbf{1} & 0 & 1 & 0 & 0 \end{bmatrix}}$$

There are two row swaps (rows 1 and 5, and rows 2 and 4), so the sign of this permutation is $(-1)^2 = 1$. Hence the determinant is $(-1)^2 \cdot 1^5 = 1$.

(b) We may generalize from part (a) above, which had exactly one nonzero term in the combinatorial definition. That matrix had 4 zeros in the third row, 3 zeros in the fifth row, 2 zeros in the first row, one zero in the fourth row, and no zeros in the second row. This pattern is the best possible for an $n \times n$ matrix: one row with $n - 1$ zeros, the next with $n - 2$, and so on. hence the smallest number of zeros in such a matrix is

$$(n-1) + (n-2) + (n-3) + \cdots + 1 = \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}.$$

19.3 (✠3.02) Let A be a 3×3 matrix. Suppose that $\det(A) = k$.

(a) Use the multilinearity property of the determinant to compute $\det(A + A)$.

(b) Use the multilinearity property of the determinant to compute $\det(-A)$.
Hint: Use the fact that $-A = A - 2A$.

(c) Explain how the result for part (b) would be different if A was a 2×2 matrix.

(a) The multilinearity property (Proposition 19.8 in the lecture notes) states that the determinant, when viewed as a function of the rows of a matrix, is linear along each row. So we view A as consisting of three rows $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_2$, and the determinant of A as a function $\det(A) = \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_2)$ of its rows. The ith row of the matrix $A + A$ is the sum of the ith row of A and the ith row of A, so:

$$\begin{aligned}
\det(A + A) &= \det(\mathbf{r}_1 + \mathbf{r}_1, \mathbf{r}_2 + \mathbf{r}_2, \mathbf{r}_3 + \mathbf{r}_3) && \text{(definition)} \\
&= \det(\mathbf{r}_1, \mathbf{r}_2 + \mathbf{r}_2, \mathbf{r}_3 + \mathbf{r}_3) \\
&\quad + \det(\mathbf{r}_1, \mathbf{r}_2 + \mathbf{r}_2, \mathbf{r}_3 + \mathbf{r}_3) && \text{(multilinearity of row 1)} \\
&= \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 + \mathbf{r}_3) + \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 + \mathbf{r}_3) \\
&\quad + \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 + \mathbf{r}_3) + \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 + \mathbf{r}_3) && \text{(multilinearity of row 2)} \\
&= \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
&\quad + \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
&\quad + \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
&\quad + \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) && \text{(multilinearity of row 3)} \\
&= 8 \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
&= 8 \det(A).
\end{aligned}$$

(b) We use the hint that $-A = A - 2A$ and the use the multilinearity property. The ith row of $A - 2A$ is the ith row of A minus twice the ith row of A, so

$$\begin{aligned}
\det(-A) &= \det(A - 2A) \\
&= \det(\mathbf{r}_1 - 2\mathbf{r}_1, \mathbf{r}_2 - 2\mathbf{r}_2, \mathbf{r}_3 - 2\mathbf{r}_3) && \text{(definition)} \\
&= \det(\mathbf{r}_1, \mathbf{r}_2 - 2\mathbf{r}_2, \mathbf{r}_3 - 2\mathbf{r}_3) \\
&\quad - 2\det(\mathbf{r}_1, \mathbf{r}_2 - 2\mathbf{r}_2, \mathbf{r}_3 - 2\mathbf{r}_3) && \text{(multlinearity of row 1)} \\
&= \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 - 2\mathbf{r}_3) - 2\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 - 2\mathbf{r}_3) \\
&\quad - 2\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 - 2\mathbf{r}_3) + 4\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 - 2\mathbf{r}_3) && \text{(multlinearity of row 2)} \\
&= \det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) - 2\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
&\quad - 2\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + 4\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
&\quad - 2\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + 4\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
&\quad + 4\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) - 8\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) && \text{(multlinearity of row 3)} \\
&= (1 - 2 - 2 + 4 - 2 + 4 + 4 - 8)\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
&= -\det(A).
\end{aligned}$$

(c) If A was a 2×2 matrix, we would find that $\det(-A) = \det(A)$, which may be found by direct computation, as $ad - bc = (-a)(-d) - (-b)(-c)$.