Homework 1 - Solutions

Introduction to Linear Algebra

Material from Lecture 1

Due Thursday, January 12, 2023

1.3 (✠1.01, 1.02) Let $\mathbf{v} \in \mathbf{R}^3$ be non-trivial, and let $\mathbf{w}, \mathbf{z} \in \mathbf{R}^3$ be non-trivial vectors both perpendicular to $\mathbf{v}$. Show that the halfway point between $\mathbf{w}$ and $\mathbf{z}$ is also perpendicular to $\mathbf{v}$.

Idea: Two vectors being perpendicular means their dot product is zero. Given that $\boldsymbol{v} \bullet \boldsymbol{w} = 0$ *and* $\boldsymbol{v} \bullet \boldsymbol{z} = 0$*, we want to show that the dot product of* $\boldsymbol{v}$ *with the halfway point between* $\boldsymbol{w}$ *and* $\boldsymbol{z}$ *is zero.*

Solution: Since $\mathbf{w} = (w_1, w_2, w_3)$ and $\mathbf{z} = (z_1, z_2, z_3)$ are perpendicular to $\mathbf{v} = (v_1, v_2, v_3)$, we have that

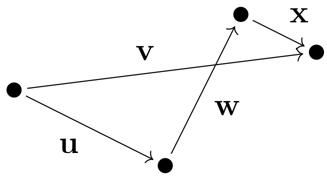
$$\begin{aligned} 0 &= \mathbf{v} \bullet \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3, \\ 0 &= \mathbf{v} \bullet \mathbf{z} = v_1 z_2 + v_2 z_2 + v_3 z_3. \end{aligned}$$

The halfway point between $\mathbf{w}$ and $\mathbf{z}$ is $\mathbf{h} = \frac{1}{2}(\mathbf{w} + \mathbf{z}) = \left(\frac{w_1+z_1}{2}, \frac{w_2+z_2}{2}, \frac{w_3+z_3}{2}\right)$, and for this vector

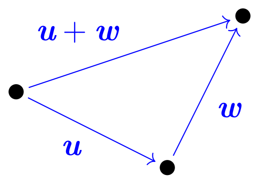
$$\begin{aligned} \mathbf{v} \bullet \mathbf{h} &= (v_1, v_2, v_3) \bullet \left(\frac{w_1 + z_1}{2}, \frac{w_2 + z_2}{2}, \frac{w_3 + z_3}{2}\right) \\ &= v_1 \bullet \frac{w_1 + z_1}{2} + v_2 \bullet \frac{w_2 + z_2}{2} + v_3 \bullet \frac{w_3 + z_3}{2} \\ &= \frac{1}{2}\left(v_1 w_1 + v_1 z_1 + v_2 w_2 + v_2 z_2 + v_3 w_3 + v_3 z_3\right) \\ &= \frac{1}{2}\left((v_1 w_1 + +v_2 w_2 + v_3 w_3) + (v_1 z_1 + v_2 z_2 + v_3 z_3)\right) \\ &= \frac{1}{2}\left(\mathbf{v} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{z}\right) \\ &= \frac{1}{2}(0 + 0) \\ &= 0. \end{aligned}$$

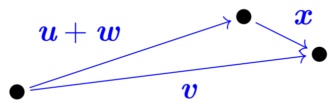
That is, $\mathbf{v} \bullet \mathbf{h} = 0$, which means that $\mathbf{v}$ and $\mathbf{h}$ are perpendicular, as desired.

1.9 (✠1.01) Use the triangle inequality to show that vector $\mathbf{v}$ is shorter than the sum of the lengths of the vectors $\mathbf{u}, \mathbf{w}, \mathbf{x}$. That is, show with the triangle inequality that $\|\mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{w}\| + \|\mathbf{x}\|$.



Idea: Combine two triangle inequalities into one inequality.





$\|\boldsymbol{u}+\boldsymbol{w}\| \leqslant \|\boldsymbol{u}\| + \|\boldsymbol{w}\|$ $\qquad$ $\|\boldsymbol{v}\| \leqslant \|\boldsymbol{u}+\boldsymbol{w}\| + \|\boldsymbol{x}\|$

In the second inequalty, replace $\|\boldsymbol{u}+\boldsymbol{w}\|$ *with the first inequality.*

Solution: Note that another way to write $\mathbf{v}$ is $\mathbf{u}+\mathbf{w}+\mathbf{x}$. By the triangle inequality on the vectors $(\mathbf{u}+\mathbf{w})$ and $\mathbf{x}$, we have

$$\|\mathbf{v}\| = \|(\mathbf{u}+\mathbf{w})+\mathbf{x}\| \leqslant \|\mathbf{u}+\mathbf{w}\| + \|\mathbf{x}\|.$$

By the triangle inequality on the vectors $\mathbf{u}+\mathbf{w}$, we have

$$\|\mathbf{u}+\mathbf{w}\| \leqslant \|\mathbf{u}\| + \|\mathbf{w}\|.$$

Putting these two lines together, we get that

$$\|\mathbf{v}\| \leqslant \|\mathbf{u}+\mathbf{w}\| + \|\mathbf{x}\| \leqslant \|\mathbf{u}\| + \|\mathbf{w}\| + \|\mathbf{x}\|,$$

as desired.