Introduction to Linear Algebra

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Preface

These notes were created to accompany the course *Introduction to Linear Algebra* for the BITL program at RTU Riga Business School. They have been used in the Fall 2021 and Spring 2022 semesters. The text may contain mistakes - please send any you find to `janis.lazovskis@rbs.lv`.

This course broadly follows Gilbert Strang's *Introduction to Linear Algebra*. You are encouraged to read the Preface to the textbook, available at `math.mit.edu/linearalgebra` before the first lecture.

Throughout the text, there are highlighted **Definitions** in green, **Inquiries** in blue, and **Algorithms** in red. The definitions are meant as key points that should be understood, if nothing else. The inquiries are meant as guiding questions to connect and unify ideas. The algorithms are meant as step-by-step instructions for complicated ideas.

At the end of each lecture there are exercises, with some solution provided at the end of the text. Exercises which require the use of a computer are marked with the symbol $\bowtie$.

Contents

Part I
Row reduction

Lecture 1: Vectors

Chapters 1.1 and 1.2 in Strang's "Linear Algebra"

- Fact 1: The dot product of a vector with itself is the square of its length.
- Fact 2: A plane in $\mathbf{R}^3$ is defined by an equation in x, y, z.

- Skill 1: Add vectors, multiply them by scalars, take their dot products.
- Skill 2: Compute the angle between vectors.

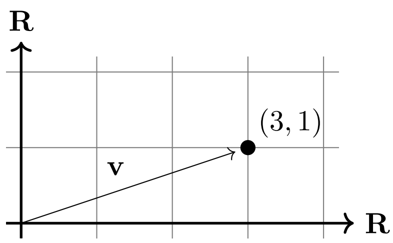
The first week will be a review of material you have seen before, but the setting may be broader, with different emphasis, and with different examples.

1.1 The algebra of vectors

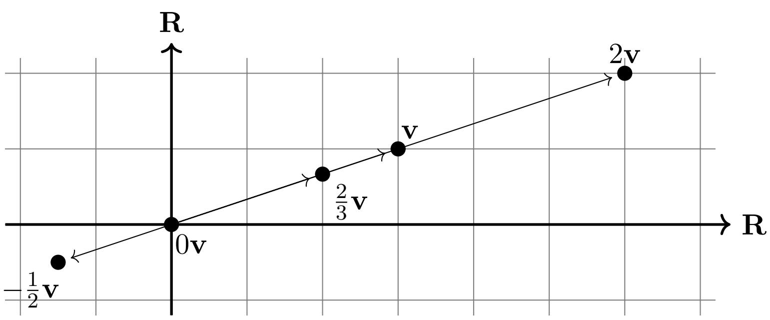
Definition 1.1: Let $n \in \mathbf{N}$. A *vector* in $\mathbf{R}^n$ is an ordered set of n elements.

The *zero vector*, or a *trivial vector*, denoted 0, is vector for which all elements are 0. Vectors that are not the zero vector are called *nontrivial*. A vector is usually thought of as a column of numbers, or a point in n-dimensional space, or the arrow to that point. All notions of a vector will be used interchangeably.

Example 1.2. The vector $\mathbf{v} = \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$ in $\mathbf{R}^2$ can also be thought of as the arrow to $(3, 1)$ or simply the point $(3, 1)$ itself.

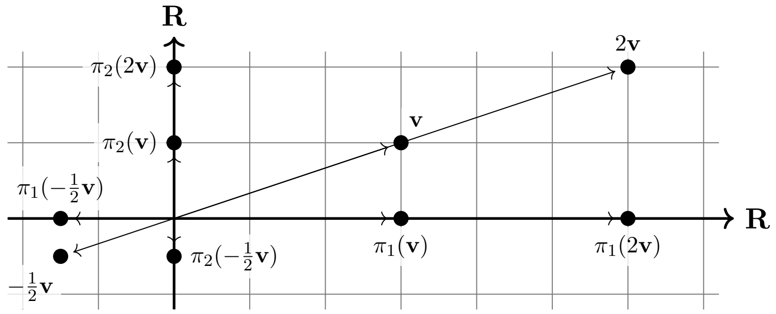


Multiplying the vector by elements of $\mathbf{R}$ we get other vectors "going in the same direction" as $\mathbf{v}$.



Definition 1.3: The numbers $v_1, \ldots, v_n \in \mathbf{R}$ in the vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbf{R}^n$ are called the *components* of the vector $\mathbf{v}$. For each component v_i, there is a unique function $\pi_i \colon \mathbf{R}^n \to \mathbf{R}$ called the *projection*, with $\pi_i(\mathbf{v}) = v_i$.

Example 1.4. Projection and multiplication by a number can be rearranged.



That is, $\pi_i(c\mathbf{v}) = c \cdot \pi_i(\mathbf{v})$ for all real numbers c and indices i. We will consider projections in more detail in Lecture 10.

Vectors are combined together in *linear combinations*.

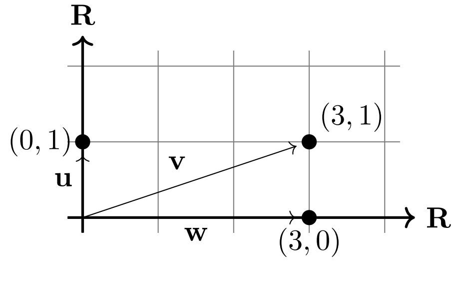
Definition 1.5: A *linear combination* of vectors is a vector $\mathbf{v} \in \mathbf{R}^n$ when it is expressed as a sum of other vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k \in \mathbf{R}^n$, and *scalars* $a_1, a_2, \ldots, a_k \in \mathbf{R}$ multiplying them. That is,

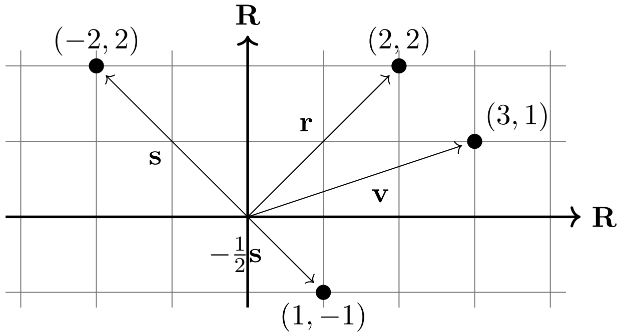
$$\mathbf{v} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_k\mathbf{w}_k.$$

When $k = 1$, the linear combination of one vector $a_1\mathbf{w}_1$ is called a *multiple* of the vector $\mathbf{w}_1$.

Example 1.6. Every vector in the plane is a linear combination of (at most) two vectors, representing the x-direction and y-direction.

$$\mathbf{v} = \mathbf{w} + \mathbf{u} = \begin{bmatrix}3\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3+0\\0+1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} \qquad \mathbf{v} = \mathbf{r} - \tfrac{1}{2}\mathbf{s} = \begin{bmatrix}2\\2\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}2 - \frac{1}{2}\cdot(-2)\\ 2 - \frac{1}{2}\cdot 2\end{bmatrix} = \begin{bmatrix}2+1\\2-1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$





The entries of vectors, and the numbers multiplying them, do not need to be numbers - they simply need to be elements of a *field*, denoted $\mathbf{F}$ in general. Unless otherwise noted, we will always use the field $\mathbf{R}$.

Example 1.7. Some common examples of fields are $\mathbf{Q}, \mathbf{R}, \mathbf{C}$.

- The set $\mathbf{N}$ is not a field because although $1 \in \mathbf{N}$, there is no $x \in \mathbf{N}$ for which $1 + x = 1$ (the *additive identity* does not exist).
- The set $\mathbf{Z}$ is not a field because although $2 \in \mathbf{Z}$, there is no number $x \in \mathbf{Z}$ for which $2x = 1$ (*multiplicative inverses* do not exist).

Definition 1.8: The *dot product*, or *inner product* of two vectors $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbf{R}^n$ is the real number $\mathbf{v} \bullet \mathbf{w} := v_1w_1 + \cdots + v_nw_n \in \mathbf{R}$.

In other words, the dot product is a function $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$.

Inquiry 1.9: Consider the vectors $\mathbf{v} = (1, 3, -1)$ and $\mathbf{w} = (2, 2, 0)$ in $\mathbf{R}^3$.

- Compute the dot products $\mathbf{v} \bullet \mathbf{w}, \mathbf{v} \bullet (2\mathbf{w})$, and $\mathbf{v} \bullet (3\mathbf{w})$. What will be $\mathbf{v} \bullet (c\mathbf{w})$, for any real number c?
- Compute the projections $\pi_i(\mathbf{v} + \mathbf{w})$ for $i = 1, 2, 3$. Do there exist vectors $\mathbf{x}, \mathbf{y}$ with $\pi_i(\mathbf{x} + \mathbf{y}) \neq \pi_i(\mathbf{x}) + \pi_i(\mathbf{y})$?
- Give an alternative definition of the dot product using the projection maps π_i.

1.2 The geometry of vectors

A key idea of vectors and their linear combinations that that they *fill a part of the space* in which they reside. The "part" of the space is another space itself.

Definition 1.10: A *plane* in $\mathbf{R}^3$ is all the points $(x, y, z) \in \mathbf{R}^3$ that satisfy an equation $ax + by + cz = d$, for some $a, b, c, d \in \mathbf{R}$. A *line* in $\mathbf{R}^3$ is all the points in $\mathbf{R}^3$ that are in two different planes that intersect.

We are often interested in planes that go through the origin $(0, 0, 0)$. They have $d = 0$ for their defining equation.

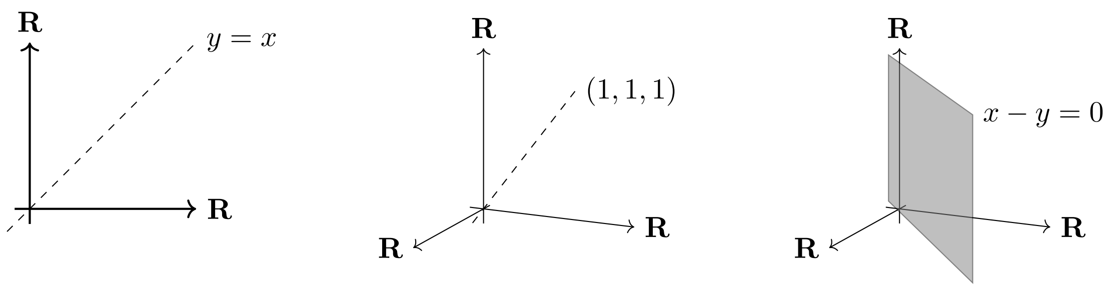
Example 1.11. Linear combinations can be described geometrically. For example:

- Linear combinations of $(1, 1)$ and $(0, 0)$ form the line $y = x$ in the plane $\mathbf{R}^2$
- Multiples of $(1, 1, 1)$ form a line in $\mathbf{R}^3$
- Linear combinations of $(1, 1, 1)$ and $(1, 1, 0)$ form the plane $x - y = 0$ in $\mathbf{R}^3$
- Linear combinations of $(1, 1, 1)$, $(1, 1, 0)$, and $(0, 1, 1)$ fill all of $\mathbf{R}^3$. For example,

$$\begin{bmatrix} 7 \\ 9 \\ -5 \end{bmatrix} = -7 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 14 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

- Linear combinations of $(1, 1, 1)$, $(1, 1, 0)$, $(0, 1, 1)$, and $(1, 0, 1)$ still fill all of $\mathbf{R}^3$. For example,

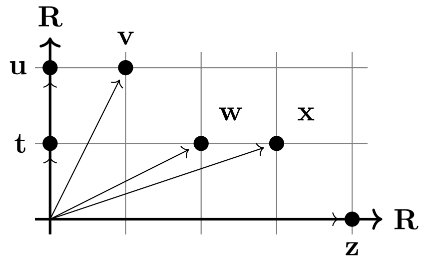
$$\begin{bmatrix} 7 \\ 9 \\ -5 \end{bmatrix} = -7 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 14 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 13 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$



Inquiry 1.12: Consider the vector $\mathbf{v} = (1, 1, 1)$ and the plane P defined by $x + y + z = 3$.

- It is clear that the plane defined by $2x + 2y + 2z = 6$ is the same as P. In general, given two planes defined by $a_1x+b_1y+c_1z = d_1$ and $a_2x+b_2y+c_2z = d_2$, how can you tell just from these equations that they are "different"?
- The point $\mathbf{v}$ lies on the plane P but $0 = (0, 0, 0)$ does not. Can you find two different planes that contain both $(1, 1, 1)$ and $(0, 0, 0)$?

Example 1.13. Consider the following vectors in $\mathbf{R}^2$.



Even though we have drawn five different vectors, there are several relationships among them:

$$\mathbf{v} = \mathbf{u} + \frac{1}{4}\mathbf{z}, \qquad \mathbf{z} = 2\mathbf{w} - \mathbf{u}, \qquad \mathbf{z} + \mathbf{v} = \mathbf{w} + \mathbf{x}.$$

These are not the only ones - there are many more.

Inquiry 1.14: In Example 1.13, the relationships given were among three or four vectors.

- Can any three of the vectors given there be related by an equation? What about any two? Don't use something trivial like $0\mathbf{u} = 0\mathbf{v} + 0\mathbf{w}$!
- Explain why every equation with four vectors (such as $\mathbf{z} + \mathbf{v} = \mathbf{w} + \mathbf{x}$) is made up of two "smaller" equations with three vectors.
- Suppose you are given three vectors in $\mathbf{R}^2$. How can you know if there is a relationship between them?

Definition 1.15: The dot product of a vector $\mathbf{v}$ with itself is the square of the *norm*, or *length*, or *distance* of the vector $\mathbf{v}$, denoted $\|\mathbf{v}\|$. That is,

$$\|\mathbf{v}\|^2 = \mathbf{v} \bullet \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2, \qquad \text{or} \qquad \|\mathbf{v}\| := \sqrt{\mathbf{v} \bullet \mathbf{v}}.$$

We know the inside of the square root will be nonnegative, as we are summing squares. The norm satisfies the following properties, for any $\mathbf{v} \in \mathbf{R}^n$:

- Non-negative: $\|\mathbf{v}\| \geqslant 0$
- Positive definite: $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$
- Multiplicative: $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$ for any $c \in \mathbf{R}$

These properties follow immediately from the properties of the real numbers and the definition of the norm above.

Definition 1.16: A vector $\mathbf{v} \in \mathbf{R}^n$ is a *unit vector* if $\|\mathbf{v}\| = 1$.

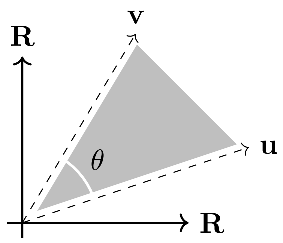
Proposition 1.17. For any $\mathbf{u}, \mathbf{v}$ nonzero in $\mathbf{R}^n$:

1. The vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.
2. The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is computed by the relation $\frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos(\theta)$
3. The *Cauchy–Schwarz inequality* holds: $|\mathbf{u}\bullet\mathbf{v}| \leqslant \|\mathbf{u}\|\|\mathbf{v}\|$
4. The *triangle inequality* holds: $\|\mathbf{u}+\mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$

Proof. To prove 1., we need to show that the norm of $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is 1. This follows as

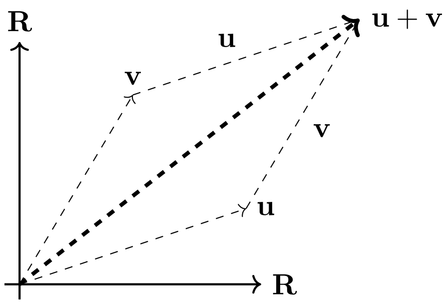
$$\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\|^2 = \frac{\mathbf{v}}{\|\mathbf{v}\|}\bullet\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|^2}(\mathbf{v}\bullet\mathbf{v}) = \frac{1}{\|\mathbf{v}\|^2}\|\mathbf{v}\|^2 = 1.$$

To prove 2., we use the law of cosines on the triangle formed by the origin 0, $\mathbf{u}$ and $\mathbf{v}$:



$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\|\mathbf{u}\|\cos(\theta)\\ (\mathbf{u}-\mathbf{v})\bullet(\mathbf{u}-\mathbf{v}) &= \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\|\mathbf{u}\|\cos(\theta)\\ \mathbf{u}\bullet\mathbf{u} - 2\mathbf{u}\bullet\mathbf{v} + \mathbf{v}\bullet\mathbf{v} &= \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\|\mathbf{u}\|\cos(\theta)\\ \frac{-2\mathbf{u}\bullet\mathbf{v}}{-2\|\mathbf{v}\|\|\mathbf{u}\|} &= \cos(\theta)\end{aligned}$$

To prove 3., use the fact that $\cos(\theta) \leqslant 1$, then take the absolute value of the equation from part 2. To prove 4., we can either draw a parallelogram and notice that the diagonal is $\mathbf{u}+\mathbf{v}$, and that it is shorter than the sum of the sides, which are $\mathbf{u}$ and $\mathbf{v}$. Or we can use algebra and part 3.



$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\bullet(\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\bullet\mathbf{u} + 2\mathbf{u}\bullet\mathbf{v} + \mathbf{v}\bullet\mathbf{v}\\ &\leqslant \|\mathbf{u}\|^2 + 2|\mathbf{u}\bullet\mathbf{v}| + \|\mathbf{v}\|^2\\ &\leqslant \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2\\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2\end{aligned}$$

□

As a result of part 2. of the proof above, if $\mathbf{u}$ is perpendicular to $\mathbf{v}$, then $\theta = \pi/2$, and so $\cos(\theta) = 0$. That is, $\mathbf{u}$ is perpendicular to $\mathbf{v}$ if and only if $\mathbf{u}\bullet\mathbf{v} = 0$.

Definition 1.18: Two non-zero vectors $\mathbf{v}$, $\mathbf{w}$ are *parallel* if there exists $c \in \mathbf{R}_{\neq 0}$ with $\mathbf{v} = c\mathbf{w}$. If $c = 1$, then the two vectors are *colinear*. In the opposite case, when the dot product $\mathbf{v}\bullet\mathbf{w} = 0$, the vectors are called *perpendicular*, or *orthogonal*.

Sometimes "parallel" is used when $c > 0$ and "anti-parallel" for $c < 0$. We will see orthogonality later in Lecture 9.

1.3 Exercises

Exercise 1.1. Consider the four vectors $\mathbf{v} = \begin{bmatrix}0\\6\\-1\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}-3\\-4\\-5\end{bmatrix}$, $\mathbf{z} = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}-5\\5\\-4\end{bmatrix}$. Find $a, b, c \in \mathbf{R}$ with $a\mathbf{v} + b\mathbf{w} + c\mathbf{z} = \mathbf{y}$.

Exercise 1.2. Check that the dot product from Definition 1.1 is *distributive* over vector addition. That is, show that $\mathbf{v} \bullet (\mathbf{u} + \mathbf{w}) = \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{w}$, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^n$.

Exercise 1.3. Let $\mathbf{v} = (1, 1, 1)$, $\mathbf{w} = (2, -2, 0)$ and $\mathbf{z} = (-3, 1, 2)$ be vectors in $\mathbf{R}^3$.

1. Using a linear equation in three variables, describe the plane of points $\mathbf{R}^3$ that are equidistant from $\mathbf{v}$ and $\mathbf{w}$.

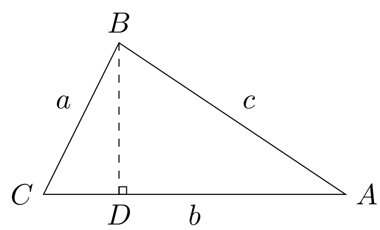
2. Using two equations, describe the line of points in $\mathbf{R}^3$ that are equidistant from $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$. Hint: A line is the intersection of two planes.

Exercise 1.4. Let S be the subset $[-5, 5] \times [-3, 3] \subseteq \mathbf{R}^2$.

1. Identify all the points in S that correspond to linear combinations $a \left[\begin{smallmatrix}3\\0\end{smallmatrix}\right] + b \left[\begin{smallmatrix}-1\\2\end{smallmatrix}\right]$, for $a, b \in \mathbf{Z}$.

2. Which of the points from part (a) lie a distance of more than 2 but less than 3 from the origin?

Exercise 1.5. The proof of Proposition 1.17 used the "law of cosines", which itself was not proved, so we prove it here. Consider the triangle below:



1. Find the formulas for $\cos(C)$ and $\sin(C)$ in the triangle BCD.

2. Rewrite $\cos(C)$ from above so it has the number $b = AC$. Use the fact that $CD = AC - b$.

3. Express the Pythagorean theorem of triangle ABD.

4. Replace the sides from part (c) with the formular from parts (a) and (b). Simplify to get the law of cosines.

Exercise 1.6. Let $\mathbf{v} \in \mathbf{R}^3$ be non-trivial, and let $\mathbf{w}, \mathbf{z} \in \mathbf{R}^3$ be non-trivial vectors perpendicular to $\mathbf{v}$. Show that the halfway point between $\mathbf{w}$ and $\mathbf{z}$ is also perpendicular to $\mathbf{v}$.

Exercise 1.7. This question is about orthogonality of vectors in Euclidean space $\mathbf{R}^n$.

1. Find $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^3$ nonzero for which $\mathbf{u}$ is perpendicular to $\mathbf{v}$, $\mathbf{v}$ is perpendicular to $\mathbf{w}$, and $\mathbf{u}$ is perpendicular to $\mathbf{w}$.

2. Find $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^3$ nonzero for which $\mathbf{u}$ is perpendicular to $\mathbf{v}$, $\mathbf{v}$ is perpendicular to $\mathbf{w}$, and $\mathbf{u}$ is colinear to $\mathbf{w}$.

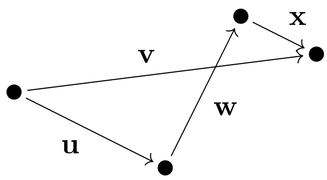
3. **Bonus:** Explain why it is not possible to have $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbf{R}^3$ nonzero with every pair of vectors orthogonal to each other.

Exercise 1.8. This question is about the Cauchy–Schwarz inequality, $|\mathbf{v} \cdot \mathbf{w}| \leqslant \|\mathbf{v}\| \cdot \|\mathbf{w}\|$.

1. Suppose that there exists $c \in \mathbf{R} \setminus \{0\}$ with $\mathbf{w} = c \cdot \mathbf{v}$. Show that the Cauchy–Schwarz inerquality holds with equality.

2. Suppose that the Cauchy–Schwarz inequality holds with equality. Show that there exists $c \in \mathbf{R} \setminus \{0\}$ with $\mathbf{w} = c \cdot \mathbf{v}$.

Exercise 1.9. Use the triangle inequality to show that vector $\mathbf{v}$ is shorter than the sum of the lengths of the vectors $\mathbf{u}, \mathbf{w}, \mathbf{x}$. That is, show with the triangle inequality that $\|\mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{w}\| + \|\mathbf{x}\|$.



Lecture 2: Matrices

Chapter 1.3 in Strang's "Linear Algebra"

- Fact 1: Matrix multiplication is associative and distributive, but not commutative
- Fact 2: Not every matrix has an inverse.

- Skill 1: Perform common operations (addition, multiplication, transpose) with matrices and vectors
- Skill 2: Multiply block matrices with each other

2.1 Types of matrices

Definition 2.1: Let $m, n \in \mathbf{N}$. An $m \times n$ *matrix* over $\mathbf{R}$ is an ordered set M of $m \cdot n$ elements.

- The space of all $m \times n$ matrices over $\mathbf{R}$ is denoted $\mathcal{M}_{m\times n}(\mathbf{R})$ or simply $\mathcal{M}_{m\times n}$, when the field is not relevant or clear from context.
- The size, or dimensions of a matrix, is the pair (m, n). By convention, the number of rows comes first.

The elements of a matrix are called its *entries*. The entry in row i, column j is called the ij-entry.

Comparing Definition 2.1 with Definition 1.1, we see that a vector in $\mathbf{R}^n$ is just a $n \times 1$ (or $1 \times n$) matrix. Similarly to vectors, the elements of matrices may be over other fields, not necessarily $\mathbf{R}$. Two matrices of particular importance are the *zero matrix* 0 (all entries are zero) and the *identity matrix* I (all entries are zero except the diagonal, which is all 1's), given by

$$0 := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad I := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The identity matrix is square, but the zero matrix does not have to be square. Sometimes to emphasize the size of the matrix, we write 0_n and I_n for matrices with n rows and n columns. For an $m \times n$ matrix A, the entry in row i and column j is denoted A_{ij} or $(A)_{ij}$ or $A(i, j)$ or a_{ij}. That is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Sometimes instead of given specfic numbers for constructing a matrix, you are given other matrices.

Definition 2.2: A matrix $M \in \mathcal{M}_{m\times n}$ is a *block matrix* if its entries are matrices instead of numbers

Example 2.3. For example, if $A \in \mathcal{M}_{2\times 3}$, $B \in \mathcal{M}_{2\times 5}$, $C \in \mathcal{M}_{3\times 3}$, and $D \in \mathcal{M}_{3\times 5}$, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{M}_{5\times 8} \qquad \text{and} \qquad \begin{bmatrix} C & 0 \\ I & D \end{bmatrix} \in \mathcal{M}_{6\times 8}$$

are both block matrices. The identity I and zero 0 matrices are used without specifying their size as blocks in a block matrix. As before, the matrix I will always be square, but 0 can be any shape.

Finally, there are three special types of square matrices:

$$\begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ * & * & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}$$

upper triangular matrix	*lower triangular matrix*	*diagonal matrix*
$a_{ij} = 0$ if $i > j$	$a_{ij} = 0$ if $i > j$	$a_{ij} = 0$ if $i \neq j$

The symbol "$*$" represents any number, and they do not all have to be the same. The two on the left are called *triangular matrices*. We will see several times over why these are special.

2.2 Operations on matrices

Definition 2.4: There are several common matrix operations.

- *sum*: the sum of $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{m\times n}$ has ij-entry $(A+B)_{ij} = A_{ij} + B_{ij}$
- *product*: the product of $A \in \mathcal{M}_{m\times n}$ and $C \in \mathcal{M}_{n\times m}$ has ij-entry $(AB)_{ij} = \sum_{k=1}^n A_{ik}B_{kj}$
- *Hadamard product*: the Hadamard product, or entry-wise product, of $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{m\times n}$ has ij-entry $(A \circ B)_{ij} = A_{ij}B_{ij}$

Example 2.5. In the special case that a matrix $A \in \mathcal{M}_{m\times n}$ is being mutiplied by a vector $\mathbf{x} \in \mathbf{R}^n$, we will have that the result $A\mathbf{x}$ will be a vector in $\mathbf{R}^m$, with $(A\mathbf{x})_i = \sum_{j=1}^m A_{ij}x_j$. For example,

$$\begin{bmatrix} 2 & 3 & -1 \\ 8 & -2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^2 A_{1j}x_1 \\ \sum_{j=1}^2 A_{2j}x_2 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 + 3\cdot 1 + (-1)\cdot(-2) \\ 8\cdot 3 + (-2)\cdot 1 + 0\cdot(-2) \end{bmatrix} = \begin{bmatrix} 11 \\ 22 \end{bmatrix}.$$

Remark 2.6. Matrix addition has the following properties, for A, B, C are matrices of the appropriate size, $c \in \mathbf{R}$, and $\mathbf{x}$ a vector:

- addition is *commutative*: $A + B = B + A$
- addition is *assocative*: $A + (B + C) = (A + B) + C$
- multiplication by a number is *distributive* over addition: $c(A + B) = cA + cB$
- multiplication by a matrix is *distributive* over addition: $C(A+B) = CA+CB$ and $(A+B)C = AC + BC$
- multiplication by a matrix or vector is *assocative*: $A(BC) = (AB)C$ and $A(B\mathbf{x}) = (AB)\mathbf{x}$

Multiplication of matrices is not always *commutative*: $AB \neq BA$.

Example 2.7. The identity (also called the *multiplicative identity*) and zero (also called the *additive identity*) matrices have special properties with addition and multiplication. For any $A \in \mathcal{M}_{m\times n}$:

- the product of A with I is A itself: $AI = IA = I$
- the product of A with 0 is is 0: $A0 = 0A = 0$

- the sum of A and 0 is A itself: $A + 0 = 0 + A = A$

In the second property, the zero matrix 0 does not have the same size every time it is used.

Remark 2.8. When multiplying block matrices, extra care has to be taken with non-commutativity. For example, if A, B, C are matrices, then

$$\begin{bmatrix} A & I \\ B & C \end{bmatrix} \begin{bmatrix} I & C \\ D & D \end{bmatrix} = \begin{bmatrix} A + D & AD + D \\ B + CD & BC + CD \end{bmatrix}.$$

The lower right entry cannot be simplified as $C(B + D)$, because it is not always true that $BC = CB$.

Definition 2.9: Let A be an $m \times n$ matrix. The *transpose* of A is written A^T, and has ij-entry $(A^T)_{ij} = A_{ji}$.

The transpose plays well with matrix operations:

$$\begin{aligned} (A + B)^T &= A^T + B^T, \\ (A\mathbf{x})^T &= \mathbf{x}^T A^T, \\ (AB)^T &= B^T A^T. \end{aligned}$$

These results follow from how the sum and product were defined in Definition 2.4.

Definition 2.10: Let A be an $n \times n$ matrix. The *inverse* of A is a matrix B for which $AB = BA = I$.

Note that the inverse of a matrix A does not always exist. When it does, it is usually denoted A^{-1}. As a result of the first property from Example 2.7, the inverse of the identity matrix is itself: $II = I$, so $I^{-1} = I$.

Example 2.11. If $A \in \mathcal{M}_{n\times n}$ is a diagonal matrix with nonzero entries on its diagonal, then its inverse is the same, but with reciprocals on the diagonal:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{a_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}.$$

Inquiry 2.12: Consider the diagonal matrix and its inverse from Example 2.11.

- If $A = \left[\begin{smallmatrix} 3 & d \\ 0 & -2 \end{smallmatrix}\right]$ and $d = 0$, what is A^{-1}? What if $d \neq 0$?
- Let $B = \left[\begin{smallmatrix} 3 & 0 & 0 \\ 0 & -5 & d \\ 0 & 0 & -1 \end{smallmatrix}\right]$, with $d \neq 0$. Find B^{-1}. Hint: $(B^{-1})_{ij} = 0$ iff $B_{ij} = 0$.
- Let $C = \left[\begin{smallmatrix} 3 & 0 & 0 & 0 \\ 0 & -5 & d & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{smallmatrix}\right]$ with $d \neq 0$. Find C^{-1}.
- Generalize the above example with $C_{ij} = d \neq 0$ instead of C_{23}, with the condition that $i < j$ (that is, C_{ij} is above the diagonal). What if C_{ij} is below the diagonal?

If $A \in \mathcal{M}_{m\times n}$ and $m \neq n$, then there may be a matrix $B \in \mathcal{M}_{n\times m}$ for which $AB = I$, but not necessarily $BA = I$, in which case B is called a *right inverse* of A. We will later see algorithms that compute the inverse, for now we just look at some examples.

Example 2.13. The inverse of the *difference matrix* is a *sum matrix*. That is, for

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

we have $AB = I$. Both of these matrices are triangular, or more specifically, lower triangular. These matrices get their names from what they do to a vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$:

$$A\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \end{bmatrix}, \qquad B\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 + x_1 \\ x_3 + x_2 + x_1 \\ x_4 + x_3 + x_2 + x_1 \end{bmatrix}.$$

Definition 2.14: Let $A \in \mathcal{M}_{m\times n}$ be a matrix, and $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{b} \in \mathbf{R}^m$ be vectors. The equation $A\mathbf{x} = \mathbf{b}$ is a *matrix equation*, and consists of m individual equations:

$$A\mathbf{x} = \mathbf{b} \qquad \iff \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= b_m. \end{aligned}$$

Finding the inverse of a matrix A is related to finding the solution $\mathbf{x}$ to a matrix equation $A\mathbf{x} = \mathbf{b}$. Indeed, if A has an inverse, then we immediately see that

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \quad \iff \quad A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \quad \iff \quad I\mathbf{x} = A^{-1}\mathbf{b} \quad \iff \quad \mathbf{x} = A^{-1}\mathbf{b}.$$

Inquiry 2.15: Let A be a matrix.

- Suppose you know that $A\left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix} -1 \\ -4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right]$. What is the inverse matrix A^{-1}? Hint: If $A\mathbf{x} = \mathbf{b}$, then putting $\mathbf{x}$ as the first column of a 2×2 matrix $\left[\begin{smallmatrix} \mathbf{x} & * \end{smallmatrix}\right]$, we get $A\left[\begin{smallmatrix} \mathbf{x} & * \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{b} & * \end{smallmatrix}\right]$.
- In general for $A \in \mathcal{M}_{m\times n}$, suppose that for any vector $\mathbf{b} \in \mathbf{R}^m$, you are able to find $\mathbf{x} \in \mathbf{R}^n$, which depends on $\mathbf{b}$, such that $A\mathbf{x} = \mathbf{b}$. Explain which vectors $\mathbf{b}$ you would choose to construct the inverse of the matrix A.
- Is the collection of vectors $\mathbf{b}$ from the previous part unique? Is there a minimum number of vectors? Give two different collections of vectors $\mathbf{b}$ that would work.

Example 2.16. The *cyclic matrix* C does not have an inverse. That is, there is no vector $\mathbf{x}$ for which

$$C\mathbf{x} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{a},$$

for any chosen $\mathbf{a}$. It is immediate that $\mathbf{a} = 0$ has a solution, when $x_1 = x_2 = x_3$. But it is also immediate that $\mathbf{a} = (1, 2, 3)$ is not a solution, because adding the three equations

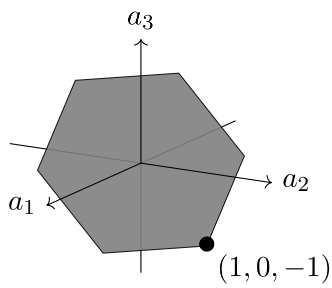
$$x_1 - x_3 = 1, \qquad x_2 - x_1 = 2, \qquad x_3 - x_2 = 3,$$

gives 0 on the left side and 6 on the left. In this situation, we say:

- when $a_1 + a_2 + a_3 = 0$, there is a solution to $C\mathbf{x} = \mathbf{a}$, or equivalently,

- all linear combinations $x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3$ lie on the plane given by $a_1 + a_2 + a_3 = 0$,

where $C = [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3]$. If we consider a_1, a_2, a_3 as changing along the x, y, z axes,respectively, we see the collection of linear combinations $x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3$ is indeed a plane:



2.3 Exercises

Exercise 2.1. A non-square matrix A may have (non-square) matrices B, C for which $AB = I$ and $CA = I$, in which case we call B a *right inverse* and C a *left inverse* for A. Let $A = \left[\begin{smallmatrix} 1 & 0 & -2 \\ 3 & -1 & 1 \end{smallmatrix}\right]$.

1. Construct a right inverse for A, that is, a 3×2 matrix B for which $AB = I$. Make it so that $BA \neq I$.

2. Try to construct a left inverse for A, that is, a 3×2 matrix C for which $CA = I$. Is it possible?

Exercise 2.2. Let A, B, C, D be $n \times n$ matrices that are invertible. Find the inverses of the following block matrices.

1. $\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}$
2. $\begin{bmatrix} I & B \\ 0 & D \end{bmatrix}$
3. $\begin{bmatrix} A & 0 \\ I & D \end{bmatrix}$

Exercise 2.3. Recall the definition of the inverse of a matrix A, which is a matrix B for which $AB = BA = I$. Show that B is *unique*. That is, show that if there exists a matrix C with $AC = CA = I$, then $C = B$.

Exercise 2.4. This question is about *triangular* matrices.

1. Show that the product of two lower triangular matrices is lower triangular.

2. Show that the product of two upper triangular matrices is upper triangular. The concept of a *transpose*, introduced in the next lecture, will make this computation easier, given your work from part (a).

3. What form will the product of a lower triangular with an upper triangular matrix have? Can you come up with an example where the result is a diagonal matrix, but the original matrices are not diagonal?

Lecture 3: Elimination

Chapters 2.1-2.4 in Strang's "Linear Algebra"

- Fact 1: Row operations are matrix multiplications.
- Fact 2: Solving a matrix equation can be understood in terms of the rows or the columns.

- Skill 1: Draw the row and column pictures for 2×2 matrix equations.
- Skill 2: Identify pivots (or their non-existence) and multipliers in matrix equations.
- Skill 3: Construct the inverse of a matrix.

This lecture reviews how to solve linear systems, and goes into more detail. Recall the three *elementary row operations*, which will be here presented as matrix multiplication:

$$\text{multiply a row by a nonzero number:} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 7 & 3 & 4 \\ 0 & 2 & -1 & 3 \\ -1 & 2 & 5 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 7 & 3 & 4 \\ 0 & 4 & -2 & 6 \\ -1 & 2 & 5 & 2 \end{array}\right]$$

$$\text{swap two rows:} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 7 & 3 & 4 \\ 0 & 2 & -1 & 3 \\ -1 & 2 & 5 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 7 & 3 & 4 \\ -1 & 2 & 5 & 2 \\ 0 & 2 & -1 & 3 \end{array}\right]$$

$$\text{add a multiple of one row to another row:} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 7 & 3 & 4 \\ 0 & 2 & -1 & 3 \\ -1 & 2 & 5 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 9 & 8 & 6 \\ 0 & 4 & -2 & 6 \\ -1 & 2 & 5 & 2 \end{array}\right]$$

The reason for interpreting these as matrix operations is to formalize the algorithm that row reduces a matrix and to build the inverse of a matrix.

3.1 The row and column pictures

The main object of study for this lecture is the matrix equation $A\mathbf{x} = \mathbf{b}$, where $A \in \mathcal{M}_{m\times n}$, $\mathbf{b} \in \mathbf{R}^n$ and $\mathbf{x}$ is a column of n variables $x_1, \dots, x_n$. You should understand this equation in two ways:

- by the *columns* of A: a linear combination of the n columns of A produces the vector $\mathbf{b}$
- by the *rows* of A: the m equations from the m rows of A describe m planes meeting at the point $\mathbf{x} \in \mathbf{R}^n$

Note that the word *plane* comes from a flat surface living in space (that is, $\mathbf{R}^3$)[1].

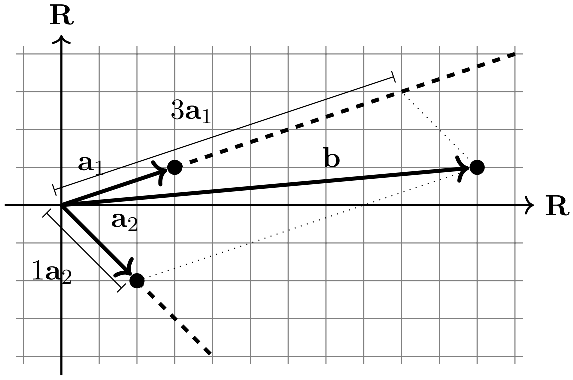
Example 3.1. Let $A = \left[\begin{smallmatrix} 3 & 2 \\ 1 & -2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{smallmatrix}\right]$ and $\mathbf{b} = \left[\begin{smallmatrix} 11 \\ 1 \end{smallmatrix}\right]$, with $\mathbf{x} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. As *columns* of A, we have a linear combination

$$x\mathbf{a}_1 + y\mathbf{a}_2 = \mathbf{b}, \qquad \text{or} \qquad x\begin{bmatrix} 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}.$$

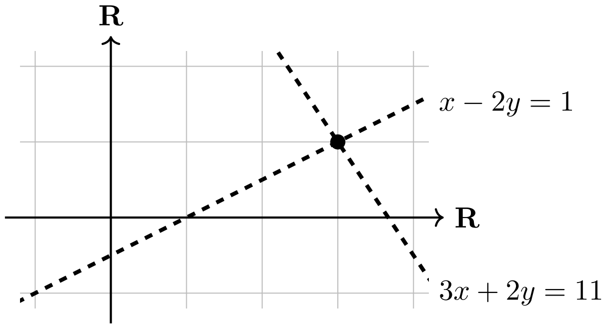
The solution to the matrix equation is the pair of coefficients x, y that satisfy the matrix equation. That is, we want to find how far along $\mathbf{a}_1$ we need to go, so that going a certain distance along $\mathbf{a}_2$ will

[1] It is more precise to say *hyperplane* to describe all the points in $\mathbf{R}^n$ satisfying a single equation. See Definition 3.4.

lead us to $\mathbf{b}$. We find a solution $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$:



As *rows* of A, we have two equations $3x + 2y = 11$ and $x - 2y = 1$, which we may interpret as lines in $\mathbf{R}^2$. This looks like the following picture (note that these are not the same lines as in the previous picture):



The two lines intersect at $(x, y) = (3, 1)$, which is the solution $\mathbf{x}$ that solves the given matrix equation $A\mathbf{x} = \mathbf{b}$. Both the column and row pictures give the same answer! This is good.

Remark 3.2. For the previous example, in the *row* picture:

- If the two lines were parallel and not colinear, there would be *no solutions*, because the lines would not intersect. For example, if instead of $3x + 2y = 11$ we had $x - 2y = -1$.
- If the lines were parallel and colinear, there would be *infinitely many solutions*, because the lines would intersect at all points. For example, if instead of $3x + 2y = 11$ we had $2x - 4y = 2$.

Inquiry 3.3: Follow the set up for drawing $A\mathbf{x} = \mathbf{b}$ from Example 3.1 for this inquiry.

- Draw the row and column pictures for $A = I \in \mathcal{M}_{2\times 2}$ and $\mathbf{b} = \left[\begin{smallmatrix} 3 \\ -4 \end{smallmatrix}\right]$. What is the solution $\mathbf{x}$?
- Draw the row and column pictures for $A = \left[\begin{smallmatrix} -1 & 1 \\ 2 & -4 \end{smallmatrix}\right]$ and $\mathbf{b} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. What is the solution $\mathbf{x}$?
- What if $A = \left[\begin{smallmatrix} -1 & 1 \\ 2 & -2 \end{smallmatrix}\right]$ for the previous point? Is there more than one solution?
- Interpret $A\mathbf{x} = 0$ having more than one solution, as a relationship between the columns (or rows) of A.

We now set up a specific algorithm (this will be the *Gaussian elimination* algorithm you may have seen earlier) for finding the solution vector $\mathbf{x}$ to a matrix equation $A\mathbf{x} = \mathbf{b}$.

Definition 3.4: Let $A\mathbf{x} = \mathbf{b}$ be a matrix equation with $A \in \mathcal{M}_{m\times n}$ and $\mathbf{b} \in \mathbf{R}^m$. The *augmented matrix* associated to this equation is the $m \times (n+1)$ matrix

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right].$$

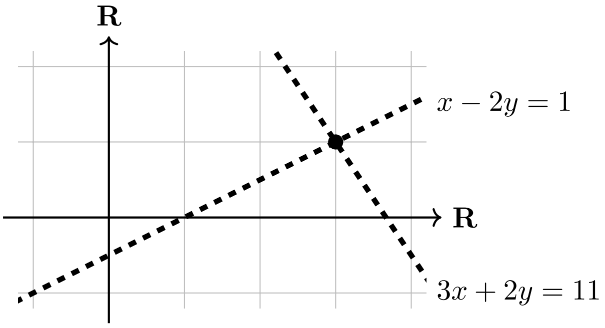
Sometimes the line separating the last two columns is not drawn. Each line $i = 1, \dots, m$ of the augmented matrix represents an equation

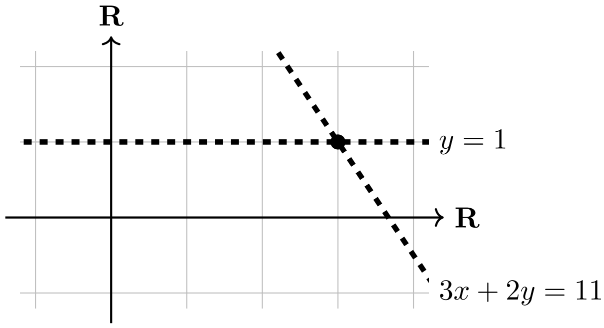
$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n = b_i$$

in n variables $x_1, \dots, x_n$ and defines a *hyperplane* in $\mathbf{R}^n$.

As you saw in Inquiry 3.3, having $A = I$ in your matrix equation makes it very easy to solve. That will be our goal now - to modify the matrix equation so that we get I instead of A. Firest, we need to make sure that this does not change the solution to the equation.

Example 3.5. Consider the augmented matrix $\left[\begin{smallmatrix} 3 & 2 & 11 \\ 1 & -2 & 1 \end{smallmatrix}\right]$ from Example 3.1. To get the first two columsn to be $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, we first will make the $(2,1)$-entry equal to zero. In the *row* picture, this means we are making the second equation flat (it will not change as x changes). The intersection of the two lines stays the same:





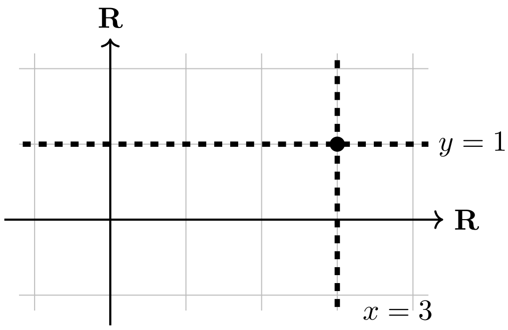
Here we added $-\frac{1}{3}$ of the first line to the second line:

$$\begin{bmatrix} 1 & 0 \\ \frac{-1}{3} & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 11 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 11 \\ 0 & -2 - \frac{2}{3} & 1 - \frac{11}{3} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 11 \\ 0 & \frac{-8}{3} & \frac{-8}{3} \end{bmatrix},$$

so technically the second equation is $-\frac{8}{3}y = -\frac{8}{3}$. Multiplying the second row by $-\frac{3}{8}$ gives the equation as we would like it to be:

$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{8} \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 11 \\ 0 & \frac{-8}{3} & \frac{-8}{3} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 11 \\ 0 & 1 & 1 \end{bmatrix}.$$

Adding -2 of the second line to the first line makes the $(2,1)$-entry 0, and makes the two lines perpendicular:



The matrix multiplication corresponding to this will give us $3x = 9$, so we simplify as well:

$$-2 \text{ times second row plus first row: } \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 11 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 9 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\frac{1}{3} \text{ times first row: } \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 9 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

We put all the matrices together from tall the steps:

$$\underbrace{\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}}_{\text{row operations}} \underbrace{\begin{bmatrix} 3 & 2 & 11 \\ 1 & -2 & 1 \end{bmatrix}}_{[A \mid \mathbf{b}]} = \underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}}_{[I \mid \mathbf{c}]}.$$

Inquiry 3.6: This inquiry is about extending Example 3.5.

- Multiply together the row operation matrices to get a matrix B. Compute BA and AB. What do you get? What can you conclude about B?
- Repeat the steps and draw the pictures for the example, but use the *column* perspective instead of the *row* perspective.

3.2 Gaussian and Gauss–Jordan elimination

We now formalize Example 3.5 into a proper *algorithm* that transforms the augmented matrix $[A \mid \mathbf{b}]$ into the augmented matrix $[I \mid \mathbf{c}]$, or at least as close as possible (it may be that some elements on the diagonal of I may be zero instead of 1):

Algorithm 1 (The Gaussian algorithm):

1. Look at the $(1,1)$-entry A_{11}.
   (a) If $A_{11} \neq 0$:
      i. Make all entries below A_{11} zero: add $-\frac{A_{21}}{A_{11}}$ of row 1 to row 2.
      ii. Add $-\frac{A_{31}}{A_{11}}$ of row 1 to row 3, and keep going until everything below A_{11} is zero.
   (b) If $A_{11} = 0$:
      i. Swap row 1 and row 2 so that the new $(1,1)$-entry is not zero, and start from the beginning.
      ii. If the first entry of row 2 is zero, swap row 1 with row 3 (or keep going down until the first element of some row is nonzero).
2. Look at the $(2,2)$-entry A_{22}.
   (a) Repeat steps (a) and (b) above with A_{22} instead, to get zeros below A_{22}.
3. Repeating this for every, the matrix A should have become upper triangular. That is, the (i,j)-entry should be 0 for $i > j$.
4. Multiply each row by the reciprocal of its first nonzero term.

Definition 3.7: The above algorithm is *Gaussian elimination*. For each row i of the augmented matrix $[A \mid \mathbf{b}]$, before any operations are done with row i,

- if $A_{ii} \neq 0$, then A_{ii} is the ith *pivot*; if $A_{ii} = 0$, then the ith pivot does not exist,
- if $A_{ii} \neq 0$, for each $k > i$, the ratio $-\frac{A_{ki}}{A_{ii}}$ is the ki-*multiplier* ℓ_{ki}.

Each step of Gaussian elimination is performed by an *elementary matrix*:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{2}{5} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{10} \end{bmatrix}$$

swaps rows 1 and 3 *permutation matrix*	subtracts $\ell_{31} = \frac{2}{5}$ times row 1 from row 3 *elimination matrix*	multiplies row 4 by $\frac{1}{10}$ *diagonal matrix*

Permutation matrices that swap rows k and i are denoted P_{ki}. Elementary matrices that subtract ℓ_{ki} times row i from row k are denoted E_{ki}.

In general, any $n \times n$ matrix that is just I with the rows rearranged is a *permuation matrix*. The steps of Gaussian elimination performed in the revrse order, starting from the bottom left and clearing zeros *above* each pivot is called *Gauss–Jordan elimination*. Together the two are simply called *elimination*.

Example 3.8. Let $A\mathbf{x} = \mathbf{b}$ be a matrix equation with $A = \left[\begin{smallmatrix} 0 & 6 & -2 \\ 4 & 8 & -4 \\ -2 & 2 & 7 \end{smallmatrix}\right]$ and $\mathbf{b} = \left[\begin{smallmatrix} 2 \\ 8 \\ 12 \end{smallmatrix}\right]$. For the associated augmented matrix, the first pivot seems to be zero, but we cannot have that, so we swap the second row with the first row. Elementary matrices are given on the right.

$\begin{bmatrix} 0 & 6 & -2 & 2 \\ 4 & 8 & -4 & 8 \\ -2 & 2 & 7 & 12 \end{bmatrix}$ 0 can not be a pivot

$\begin{bmatrix} 4 & 8 & -4 & 8 \\ 0 & 6 & -2 & 2 \\ -2 & 2 & 7 & 12 \end{bmatrix}$ swap first two rows, 4 is first pivot — previous matrix multiplied by $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 4 & 8 & -4 & 8 \\ 0 & 6 & -2 & 2 \\ 0 & 6 & 5 & 16 \end{bmatrix}$ $\dfrac{-1}{2}$ is multiplier ℓ_{31}, 6 is second pivot — previous matrix multiplied by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 4 & 8 & -4 & 8 \\ 0 & 6 & -2 & 2 \\ 0 & 0 & 7 & 14 \end{bmatrix}$ 1 is multiplier ℓ_{32}, 7 is third pivot — previous matrix multiplied by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

This is now a system $U\mathbf{x} = \mathbf{c}$, for $U = \left[\begin{smallmatrix} 4 & 8 & -4 \\ 0 & 6 & -2 \\ 0 & 0 & 7 \end{smallmatrix}\right]$ and $\mathbf{c} = \left[\begin{smallmatrix} 8 \\ 2 \\ 14 \end{smallmatrix}\right]$. The letter "$U$" is used for "upper triangular". We then have three equations:

$$\begin{aligned} 4x + 8y - 4z &= 8, \\ 6y - 2z &= 2, \\ 7z &= 14. \end{aligned}$$

To find the vector $\mathbf{x}$ which solves this system, we can continue with Gauss–Jordan elimination, or we can use back substitution from the bottom row up to find $z = 2$, $y = 1$, $x = 2$.

Remark 3.9. As observed in Inquiry 3.6, the elementary matrices together form the inverse. Below are some common inverses.

- The inverse of a 2×2 matrix exists if and only if $ad - bc \neq 0$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- The inverse of a diagonal matrix exists iff the entries on the diagonal are nonzero:

$$\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 & \cdots & 0 \\ 0 & 1/d_2 & 0 & \cdots & 0 \\ 0 & 0 & 1/d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

- Similarly, the inverse of an upper triangular matrix exists iff the entries on the diagonal are nonzero. If some are zero, it immediately means we are missing some pivots (as everything below the diagonal is zero).

Taking the inverse of a product of matrices reverses their order: $(AB)^{-1} = B^{-1}A^{-1}$. This follows as

$$\begin{aligned} AB(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} && \text{(commutativity of multiplication)} \\ &= AIA^{-1} && \text{(definition of inverse)} \\ &= (AI)A^{-1} && \text{(commutativity of multiplication)} \\ &= AA^{-1} && \text{(property of identity matrix)} \\ &= I && \text{(definition of inverse)} \end{aligned}$$

Example 3.10. Let $A = \left[\begin{smallmatrix} 4 & 8 & -4 \\ 0 & 6 & -2 \\ -2 & 2 & 1 \end{smallmatrix}\right]$, for which we want to find the inverse. To do this, we work with the block matrix $[A\ I]$, and on it we do not only Gaussian elimination on the matrix, as in Example 3.8, but also Gauss–Jordan elimination, which clears the matrix above the pivots. Elementary matrices are given on the left.

$$\begin{bmatrix} 4 & 8 & -4 & 1 & 0 & 0 \\ 0 & 6 & -2 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{4 is first pivot}$$

$$\begin{bmatrix} 4 & 8 & -4 & 1 & 0 & 0 \\ 0 & 6 & -2 & 0 & 1 & 0 \\ 0 & 6 & -1 & 1/2 & 0 & 1 \end{bmatrix} \quad \frac{-1}{2} \text{ is multiplier } \ell_{31}\text{, 6 is second pivot} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 & -4 & 1 & 0 & 0 \\ 0 & 6 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1 \end{bmatrix} \quad \text{1 is multiplier } \ell_{32}\text{, 1 is third pivot} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 & -4 & 1 & 0 & 0 \\ 0 & 6 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1/2 & -1 & 1 \end{bmatrix} \quad \text{0 above third pivot, second line} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 & 0 & 3 & -4 & 4 \\ 0 & 6 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1/2 & -1 & 1 \end{bmatrix} \quad \text{0 above third pivot, second line} \quad \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 & 13/3 & -8/3 & 20/3 \\ 0 & 6 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1/2 & -1 & 1 \end{bmatrix} \quad \text{0 above second pivot} \quad \begin{bmatrix} 1 & -8/6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 13/12 & -2/3 & 5/3 \\ 0 & 1 & 0 & -1/6 & -1/6 & -1/3 \\ 0 & 0 & 1 & 1/2 & -1 & 1 \end{bmatrix} \quad \text{multiply by the pivot reciprocals} \quad \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have now reached the matrix $[I\ A^{-1}]$. To see the submatrix on the right is really the inverse, first multiply the elementary matrices together to get E. Above we showed that

$$E[A\ I] = [I\ B]$$

for some matrix B (which we are trying to show is the inverse of A). Block multiplication tells us that

$$E[A\ I] = [EA\ EI] = [EA\ E] \quad \Longrightarrow \quad EA = I \text{ and } E = B.$$

It follows that $BA = I$, which means that B is the inverse of A.

Remark 3.11. We now have a new, equivalent definition of $A \in \mathcal{M}_{n\times n}$ not having an inverse: If elimination of $[A\ I]$ results in $[J\ B]$, where J is almost I, but has some zeros on the diagonal, then A has no inverse.

Inquiry 3.12: This inquiry is about elimination using block matrices. Let $A, B \in \mathcal{M}_{2\times 2}$ have inverses.

- Let $C = \left[\begin{smallmatrix} A & 0 \\ 0 & I_2 \end{smallmatrix}\right]$ be a block matrix. Find the inverse 4×4 matrix C^{-1}.
- Let $D = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$ be a block matrix. Find the inverse matrix D^{-1}.
- Will $\left[\begin{smallmatrix} A & B \\ 0 & B \end{smallmatrix}\right]$ have an inverse? How do you know? What about $\left[\begin{smallmatrix} A & I_2 \\ I_2 & B \end{smallmatrix}\right]$?

3.3 Exercises

Exercise 3.1. Consider the matrix equation $A\mathbf{x} = \mathbf{b}$, given by $\left[\begin{smallmatrix} 3 & -1 & 2 \\ 6 & -2 & -1 \\ 1 & -1 & -1 \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] = \left[\begin{smallmatrix} 5 \\ -2 \\ -3 \end{smallmatrix}\right]$.

1. Do Gaussian elimination on the augmented matrix $[A \mid \mathbf{b}]$ to clear values below the diagonal.

2. Do Gauss–Jordan elimination on the result from part 1. to clear values above the diagonal. What is the solution to the equation?

3. Multiply the elementary matrices you created in parts 1. and 2. together to find the inverse matrix A^{-1}. Note that you need to multiply by diagonal matrices to make the diagonal entries be 1.

Exercise 3.2. Construct a 3×3 matrix A which has:

1. pivots 1,2,3

2. pivots 1,2,3 and multipliers $\ell_{32} = 4$, $\ell_{31} = 5$ and $\ell_{21} = 6$

3. only two pivots 1 and 2, but no zeros in any positions

Exercise 3.3. Let A be a 3×3 matrix.

1. Find the pivots when A has each of the following forms. The numbers $a, \ldots, i$ are all nonzero.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & bd/a & f \\ d & bd/a & i \end{bmatrix} \qquad \begin{bmatrix} 0 & b & c \\ 0 & e & ce/b \\ 0 & e & ce/b \end{bmatrix}$$

all pivots — no first pivot — no second pivot — no first or third pivot

⋈ 2. Write a function that takes in such a matrix and returns a list of the three pivots. You may assume that all of the pivots exist.

⋈ 3. Run your function on 1000 random 3×3 matrices with entries in the range $[-1, 1]$. What is the range and the average of all the pivots? How often do you get a zero?

In Python, you may use consider A as a list of lists `[[a,b,c],[d,e,f],[g,h,i]]`.

Exercise 3.4. This question is about the three permutation matrix examples given in Definition **??**.

1. Is the product of all three a permutation matrix?

2. Are the inverses of each still permutation matrices?

Exercise 3.5. Suppose that $A_i \in \mathcal{M}_{n \times n}$ has an inverse A_i^{-1}, for $i = 1, \ldots, k$. What is the inverse of the k-fold product $A_1 A_2 \cdots A_k$?

Exercise 3.6. Using Gauss–Jordan elimination, find the inverse matrix of $A = \left[\begin{smallmatrix} 0 & 2 & -1 \\ 1 & 0 & -4 \\ 2 & 2 & 2 \end{smallmatrix}\right]$.

Lecture 4: Factorization

Chapters 2.5-2.7 in Strang's "Linear Algebra"

- Fact 1: Every matrix A can be decomposed as $A = LU$ into lower and upper triangular factors.
- Fact 2: Inverses of elementary matrices are elementary matrices.

- Skill 1: Decompose A and PA as LU and LDU.
- Skill 2: Identify when $A\mathbf{x} = \mathbf{b}$ has no solutions or infinitely many solutions.

This lecture is about *factorization*, or *decomposition*, for a square matrix $A \in \mathcal{M}_{n\times n}$. Similarly to factoring an integer as the product of two factors (such as $12 = 3 \cdot 4$), we will factor A as the product of two triangular matrices. We will do this in four ways:

$$A = LU, \qquad A = LDU, \qquad PA = LU, \qquad PA = LDU.$$

The matrix L is lower triangular, U is upper triangular, D is diagonal, and P is a permutation matrix. The first two ways are for matrices that do not require row swaps when doing elimination, otherwise row swaps are caputred in the permutation matrix P.

4.1 Lower and upper factors

To get the lower factor L and the upper factor U, we apply the Gaussian and Gauss–Jordan algorithms from Section 3.2. First we make an observation about the inverse of elementary matrices.

Remark 4.1. The elementary matrix E_{ki} from Gaussian elimination representing the row operation that substracts ℓ_{ki} times row i from row k is just the identity with $-\ell_{ki}$ in the (ki)-position. Its inverse is similarly the identity, but with ℓ_{ki} in the same (ki)-position:

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{2}{5} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{5} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad E_{31}E_{31}^{-1} = E_{31}^{-1}E_{31} = I.$$

The same works if $-\ell_{ki}$ is above the diagonal for Gauss–Jordan elimination (that is, $k < i$).

> **Inquiry 4.2:** Consider Remark 4.1 about the inverses of elementary matrices. Let $A \in \mathcal{M}_{4\times 4}$.
>
> - Let E_{32} be an elementary matrix with the multiplier $-\ell_{32} = 2$ in the $(3, 2)$-position. What row reduction (elimination) step does the multiplication $E_{32}A$ represent?
> - Give an example of A so that $(E_{32}A)_{32} = 0$.
> - Does the inverse matrix E_{32}^{-1} represent a row operation? If yes, which one?

Example 4.3. Let $A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$, which is eliminated as:

$$\underbrace{\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{32}} \underbrace{\begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_1}}_{\text{Gaussian elimination}}, \qquad \underbrace{\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{13}} \underbrace{\begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_1} = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_2}}_{\text{Gauss–Jordan elimination}}.$$

This is the end of elimination, because we have a diagonal matrix. The first multiplier was $\ell_{32} = 2$ and the second multiplier was $\ell_{13} = -1$. The decomposition comes from putting these two steps together and taking inverses:

$$\begin{aligned} E_{13}E_{32}A &= A_2 \\ E_{32}A &= E_{13}^{-1}A_2 \\ A &= E_{32}^{-1}E_{13}^{-1}A_2 \end{aligned} \qquad \underbrace{\begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \underbrace{\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{E_{32}^{-1}}}_{L} \underbrace{\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{13}^{-1}} \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_2}}_{U} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Remark 4.4. We make several observations about the $A = LU$ decomposition:

- The lower triangular matrix L represents the steps of Gaussian elimination, and has 1's on the diagonal.
- The upper triangular matrix U represents the steps of Gauss–Jordan elimination, and has the pivots of A on the diagonal.

Inquiry 4.5: This is about extending $A = LU$ into $A = LDU$.

- In the $A = LU$ factorization from Example 4.3, the upper triangular matrix U has numbers that are not 1's on the diagonal. Do the row reduction steps on U that make all ements on the diagonal be 1. What are the corresponding elementary matrices?
- Express U from the previous point as $U = DU'$, where D is diagonal and U' is upper triangular with 1's on the diagonal.
- Generalize the above point: If $U = \left[\begin{smallmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{smallmatrix}\right]$ with nonzero diagonal elements, decompose it as $U = DU'$.

Remark 4.6. Elimination is the same for a matrix A or an augmented matrix $[A\ \mathbf{b}]$, but the lack of pivots for the augmented matrix indicates one of two situations: if elimination produces a row with

- all zeros except the last entry: then there are *no solutions*, because it implies an equation such as $0x + 0y + 0z = 1$, or $0 = 1$.
- all zeros: then there are *inifinitely many solutions*, because we then only have $n - 1$ equations but still n unknowns, so one of the unknowns can be freely chosen.

The implication is that if we applied the elimination algorithm to just the matrix A, then we would get a row of zeros in both cases.

Definition 4.7: A matrix $A \in \mathcal{M}_{m\times n}$ is *singular* if elimination returns at least one row of zeros. If there are no zero rows after elimination, then A is *non-singular*.

Example 4.8. Consider the matrix equation from $A\mathbf{x} = \mathbf{b}$ from Example 3.1, but change it slightly: $\left[\begin{smallmatrix} 3 & 2 \\ -3 & -2 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 11 \\ 1 \end{smallmatrix}\right]$. For elimination we subtract -1 times the first row from the second row:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \left[\begin{array}{cc|c} 3 & 2 & 11 \\ -3 & -2 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 2 & 11 \\ 0 & 0 & 12 \end{array}\right].$$

In the row picture, we are looking for the intersection of $3x + 2y = 11$ and $0x + 0y = 12$, or $0 = 12$. Since $0 = 12$ is a contradiction, no solution exists. Alternatively, if we changed both A and $\mathbf{b}$ to the equation $\left[\begin{smallmatrix} 3 & 2 \\ -3 & -2 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 11 \\ -11 \end{smallmatrix}\right]$, then the same step of Guassian elimination would give a full row of zeros:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\left[\begin{array}{cc|c} 3 & 2 & 11 \\ -3 & -2 & -11 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 2 & 11 \\ 0 & 0 & 0 \end{array}\right].$$

The row picture asks for the intersection of $3x + 2y = 11$ and $0x + 0y = 0$. We quickly see that every vector $\mathbf{x} = \left[\begin{smallmatrix} x \\ \frac{1}{2}(11-3x) \end{smallmatrix}\right]$, for any $x \in \mathbf{R}$, will satisfy the equation $A\mathbf{x} = \mathbf{b}$. Hence we have infintely many solutions.

Inquiry 4.9: Consider the matrix equation $A\mathbf{x} = \mathbf{b}$, for $A = \left[\begin{smallmatrix} 3 & 2 \\ 1 & -2 \end{smallmatrix}\right]$ and $\mathbf{b} = \left[\begin{smallmatrix} 11 \\ 1 \end{smallmatrix}\right]$.

- Decompose A into its lower and upper factors LU.
- Do the same as above, but attempt to decompose the augmented matrix $[A\ \mathbf{b}]$. Can you find "upper" and "lower" factors as well?
- Interpret the lower and upper factors by the row picture, as two pairs of lines in $\mathbf{R}^2$. Is there a relationship betwen the original row picture for the complete equation $A\mathbf{x} = \mathbf{b}$? This was discussed in Example 3.1.

The elimination algorithm from Section 3.2 was made more complicated by the fact that not all pivots may exist, in which case we need to swap rows so that we do not divide by zero. We now consider this type of elimination.

4.2 Row swaps and permutation matrices

The algorithm in Section 3.2 indicated to swap rows when there are zeros in the pivot positions when we reach them. However, to get tp the desired decomposition $PA = LU$, we need to put all the matrices representing row swaps together - so every time we get to a pivot that doesn't exist (is zero), we swap rows for the original matrix, and start from the beginning.

Example 4.10. Let $A = \left[\begin{smallmatrix} 3 & 2 & -1 \\ 6 & 4 & 0 \\ 0 & 4 & 1 \end{smallmatrix}\right]$, for which Gaussian elimination begins as:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}}\underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & 0 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 4 & 1 \end{bmatrix}}_{A_1}, \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_{23}}\underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 4 & 1 \end{bmatrix}}_{A_1} = \underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{A_2}.$$

The numbers to be used for pivots are highlighted - note the problem in A_1, which we resolve by a row swap. Continuing elimination from here, we would end up with something like $EPE'A = A_n$, where E and E' are elementary matrices and P is the row swap matrix. Rearranging for A is not as nice in this case, so we apply the row swap P_{23} at the very beginning,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_{23}}\underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & 0 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & 1 \\ 6 & 4 & 0 \end{bmatrix}}_{PA},$$

and now apply the usual elimination steps.

Inquiry 4.11: Note that at the end of Gaussian elimination, you have a diagonal matrix on the left side, and you know inverses of diagonal matrices. This inquiry explores the elimination steps from Example 4.10.

- Continue the elimination algorithm from A_2 until you get a diagonal matrix. Multiply its inverse to get an inverse for A (this will be the product of the elementary matrices).
- Begin with PA instead of A, and apply the elimination algorithm to it, until you get a diagonal matrix. As before, multiply the diagonal by its inverse to get an inverse for A.
- Compare the two inverse you got for A - are they the same? Are the elementary matrices involved in construction of the inverse the same? What are the similarities?

Remark 4.12. If swapping rows does not give you enough pivots, it may be that you will get a row of zeros, as described in Example 4.8. In this case elimination will still give you the LU-decomposition, but the difference will be that you have to stop elimination before you get a diagonal matrix.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & 0 \\ -3 & -2 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 2 \\ -3 & -2 & 2 \end{bmatrix}}_{A_1}, \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 2 \\ -3 & -2 & 2 \end{bmatrix}}_{A_1} = \underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}}_{A_2},$$

and swapping the second and third rows gives us $\left[\begin{smallmatrix} 3 & 2 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{smallmatrix}\right]$, which still does not have enough pivots. However, multiplying by the inverses of the elementary matrices we applied still gives an LU-decomposition:

$$\underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & 0 \\ -3 & -2 & 2 \end{bmatrix}}_{A} = \underbrace{\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E_{31}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{-1}}}_{L} \underbrace{\underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}}_{A_2}}_{U} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}}_{U}.$$

You now have all the tools you need to decompose the matrices A and PA as LU or LDU. We finish off the lecture with some useful types of matrices.

Definition 4.13: Let A be an $n \times m$ matrix.

- The matrix A is *symmetric* if $m = n$ and $A_{ij} = A_{ji}$ for all i, j.
- The matrix A is *skew-symmetric* if $m = n$ and $A_{ij} = -A_{ji}$ for all i, j.

Observe that another way to express that A is symmetric is to say that $A = A^T$, and another way to express that A is skew-symmetric is to say $A = -A^T$. Note that if $A \in \mathcal{M}_{n\times n}$ is symmetric, then its decomposition into $A = LDU$ has $L = U^T$.

Remark 4.14. The transpose can be thought of as a function $\mathcal{M}_{m\times n} \to \mathcal{M}_{n\times m}$. As noted in Definition 2.9, if plays nicely with the addition, multiplication, and inverse functions. Moreover, the dot product of two vectors from Definition 1.1 can be thought of as matrix multiplication, if we use the transpose:

$$\mathbf{v} \bullet \mathbf{w} = \mathbf{v}^T \cdot \mathbf{w} \tag{1}$$

$$\in \mathbf{R}^n \qquad \in \mathbf{R}^n \qquad \in \mathcal{M}_{1\times n} \qquad \in \mathcal{M}_{n\times 1}$$

This is why we need to be careful with the multiplication symbol $\cdot$, always being aware of the sizes of objects we are working with. That is because multiplying the other way $\mathbf{w} \cdot \mathbf{v}^T$ gives an $n \times n$ matrix, which is called the *outer product*:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \mathbf{R}^4, \qquad \mathbf{v}^T\mathbf{w} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = 1 \cdot 1 + 2 \cdot (-1) + 3 \cdot 2 + 4 \cdot (-2) = -3 \quad \in \mathbf{R} = \mathcal{M}_{1\times 1}$$

$$\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} \in \mathbf{R}^4, \qquad \mathbf{w}\mathbf{v}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 2 & -2 & 4 & -4 \\ 3 & -3 & 6 & -6 \\ 4 & -4 & 8 & -8 \end{bmatrix} \qquad \in \mathcal{M}_{4\times 4}$$

Example 4.15. Taking the transpose of a product of a matrix with a vector is just like taking the tranpose of two matrices. Using the property from Equation (1) and the observations in Remark 4.14, we see some interesting results. For $A \in \mathcal{M}_{m\times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, we have

$$A\mathbf{x} \bullet \mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = \mathbf{x}^T(A^T\mathbf{y}) = \mathbf{x} \bullet (A^T\mathbf{y}).$$

4.3 Exercises

Exercise 4.1. Consider the matrix factorization

$$\underbrace{\begin{bmatrix} 6 & 0 & -2 \\ 1 & 3 & 4 \\ -3 & -8 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 6 & 0 & -2 \\ 0 & 2 & 13/3 \\ 0 & 0 & 113/9 \end{bmatrix}}_{U}.$$

The values a, b, c are determined by the multipliers from row operations to clear the entries below the pivots. What are these values?

Exercise 4.2. Decompose $A = \begin{bmatrix} 3 & 1 & 1 \\ 3 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}$ into $PA = LDU$ factorization.

Exercise 4.3. Decompose the matrix A from Example 3.8 as $PA = LDU$.

Exercise 4.4. Suppose that $A \in \mathcal{M}_{n\times n}$ is a product of elementary matrices, that is, $A = E_1 \cdot E_2 \cdots E_k$, where E_i is one of the three types of elementary matrices given in Definition 3.7 . Explain why A is invertible.

Part II

Vector spaces

Lecture 5: Vector spaces: column space and nullspace

Chapters 3.1 and 3.2 in Strang's "Linear Algebra"

- Fact 1: A vector space is something like $\mathbf{R}^n$.
- Fact 2: The column space and nullspace of any matrix are vector spaces.

- Skill 1: Determine if something is a vector space and subspace.
- Skill 2: Construct the column space and nullspace of a matrix as spans.

This lecture serves two purposes: to introduce the very powerful topic of *vector spaces*, and to provide two concrete examples of such vector spaces. These two examples will justify the phrase "to be in the solution space" of an equation $A\mathbf{x} = \mathbf{b}$.

5.1 Vector spaces

Recall from Lecture 2 that a *field* is a set with nice properties, such as $\mathbf{R}, \mathbf{Q}, \mathbf{C}$. Fields have addition and multiplication built into them. We now define a set that has new properties.

Definition 5.1: Let V be a set and $\mathbf{F}$ a field. The elements of $\mathbf{F}$ are called *scalars*. The set V is a *vector space* if there are two operations

- addition $+ \colon V \times V \to V$,
- scalar multiplication $\cdot : \mathbf{F} \times V \to V$,

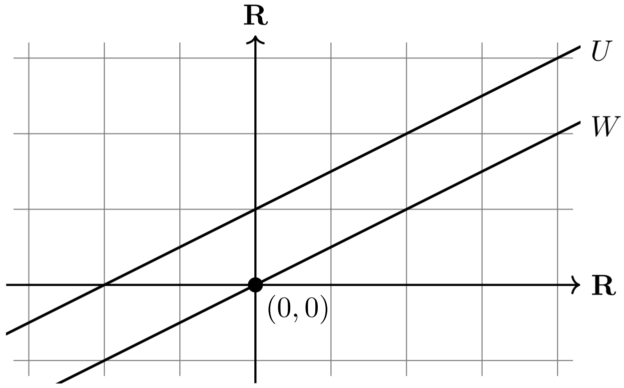
that satisfy the follow properties, for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in \mathbf{F}$:

1. addition has an identity element: there exists $0 \in V$ with $0 + \mathbf{v} = \mathbf{v}$
2. addition has inverse elements: there exists $-\mathbf{v} \in V$ with $\mathbf{v} + (-\mathbf{v}) = 0$
3. scalar multiplication has an identity element: there exists $1 \in \mathbf{F}$ with $1\mathbf{v} = \mathbf{v}$
4. addition is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
5. addition is associative: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
6. scalar multiplication is distributive over addition: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
7. scalar multiplication is distributive over field addition: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$
8. field multiplication is compatible with scalar multiplication: $(ab)\mathbf{v} = a(b\mathbf{v})$

If V is a vector space and $W \subseteq V$ is a subset of V and is a vector space on its own, with the same two operations satisfying the same properties, then W is a *subspace* of V. It is immediate that every vector space is a subspace of itself, so whenever $W \subseteq V$ is a subspace and $W \neq V$, we say W is a *proper subspace* of V.

Example 5.2. We consider some basic examples of vector spaces.

- The empty set $\emptyset$ is not a vector space, because vector space must contain the zero vector.
- The set $V = \{0\}$ is a vector space, called the *trivial* or *zero* vector space.
- The space $\mathcal{M}_{2\times 2}$ is a vector space, with addition being matrix addition, and scalar multiplication the usual scalar multiplication over $\mathbf{R}$. This space is 4-dimensional, though we will see the notion of dimension next lecture.
- For $V = \mathbf{R}^2$, the set $W = \{c(2,1) \ : \ c \in \mathbf{R}\} \subseteq V$, which is all the multiples of $\mathbf{v} = (2,1)$, is a subspace of $\mathbf{R}^2$. The set $U = \{c(2,1) + (0,1) \ : \ c \in \mathbf{R}\} \subseteq V$, which is the same as W but shifted up by 1 unit, is not a vector space, as $(0,0) \notin U$.



Inquiry 5.3: This inquiry generalizes the notion of a vector space, continuing with the last example in Example 5.2. The set U there looked like it should be a vector space - every element of U can be expressed as $\mathbf{u} = c(2,1) + (0,1)$. Define vector addition and scalar multiplication on U, for $\mathbf{F} = \mathbf{R}$, by

$$\begin{aligned} U \times U &\to U, \\ (\mathbf{u}_1, \mathbf{u}_2) &\mapsto (c_1 + c_2)(2,1) + (0,1), \end{aligned} \qquad \begin{aligned} \mathbf{R} \times U &\to U, \\ (a, \mathbf{u}_1) &\mapsto ac_1(2,1) + (0,1), \end{aligned}$$

where $\mathbf{u}_1 = c_1(2,1) + (0,1)$ and $\mathbf{u}_2 = c_2(2,1) + (0,1)$.

- Check that multiplication distributes over addition. That is, check that property 7. is satisfied.
- Find the additive identity, additive inverse, multiplicative identity on U so that properties 1.-3. are satisfied.
- Explain why U, with this vector space structure, is not a subspace of $\mathbf{R}^2$.
- Instead of $(0,1)$ at the beginning, put $(-2,0)$ in its place. What changes? Can any vector be chosen here? Which vector would you choose?

This type of structure is called an *affine space*.

Remark 5.4. We make some observations about vector spaces and subspaces.

- Every vector space and subspace must contain the zero vector.
- Any line through the origin is a subspace of $\mathbf{R}^n$.
- A subspace containing $\mathbf{u}$ and $\mathbf{v}$ must contain every linear combination $a\mathbf{u} + b\mathbf{v}$.

Example 5.5. Combining the above remark and Example 5.2, we see that $U = \{$all upper triangular matrices $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]\} \subseteq \mathcal{M}_{2\times 2}$ is a subspace of $\mathcal{M}_{2\times 2}$, as is $D = \{$all diagonal matrices $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]\} \subseteq \mathcal{M}_{2\times 2}$. Moreover, D is a subspace of U.

Definition 5.6: Let V be any vector space, such as $\mathbf{R}^n$, and $X = \{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subseteq V$ any collection of elements of V. Then the space of all *linear combinations* of elements of X, written

$$\text{span}(X) = \left\{ \sum_{i=1}^{n} c_i \mathbf{v}_i \ : \ c_i \in \mathbf{F} \right\}.$$

This space is called the *span* of the vectors in X.

With the span, we can describe a very large vector space by using a small number of vectors. Finding the smallest number of vectors will play an important role in future lectures.

Proposition 5.7. For V a vector space and $X = \{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subseteq V$, the span of X is a vector space and a subspace of V.

Proof. To see span(X) is a vector space, note that every element in span(X) is a vector in V. Adding two elements in span(X) keeps us in the span:

$$\mathbf{a} + \mathbf{b} = \sum_{i=1}^{n} a_i \mathbf{v}_i + \sum_{i=1}^{n} b_i \mathbf{v}_i = \sum_{i=1}^{n} (a_i + b_i)\mathbf{v}_i \in \text{span}(X).$$

Scalar multiplication works similarly. The identity and inverse elements are the same as in V, and clearly the zero element is in span(X), by choosing all the coefficients $c_i = 0$. Hence span$(X) \subseteq V$ is a subspace. □

Note that the above result follows immediately from Example 5.2, which said that all multiples of a single vector is a vector space, and by repeated application of Definition 5.9, which will say that $V + W$ is a vector space, for any vector spaces V, W.

Example 5.8. Two dimensional Euclidean space $\mathbf{R}^2$ can be described in several ways as a span:

- $\mathbf{R}^2 = \text{span}\left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right], \left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]\right)$ because $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \frac{x-y}{2}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] + \frac{x-y}{2}\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right] - y\left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$
- $\mathbf{R}^2 = \text{span}\left(\left[\begin{smallmatrix}3\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\4\end{smallmatrix}\right]\right)$ because $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = \frac{x}{3}\left[\begin{smallmatrix}3\\3\end{smallmatrix}\right] + \frac{3y-x}{12}\left[\begin{smallmatrix}0\\4\end{smallmatrix}\right]$.

Definition 5.9: Let V, W be two vector spaces. Their *direct sum*, or simply *sum*, is the vector space

$$V \oplus W := \{(\mathbf{v}, \mathbf{w}) \ : \ \mathbf{v} \in V, \mathbf{w} \in W\},$$

with vector addition and scalar multiplication defined component-wise. That is, $(\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2) = (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2)$ and $c(\mathbf{v}, \mathbf{w}) = (c\mathbf{v}, c\mathbf{w})$. If there exists a vector space U with $V, W \subseteq U$, then we have the vector space

$$V + W := \{\mathbf{v} + \mathbf{w} \ : \ \mathbf{v} \in V, \mathbf{v} \in W\}.$$

In this case, we have all linear combinations of vectors from both spaces. This is called the subspace *generated by* U and V. It is the smallest subspace containing $U \cup V$, which itself is not necessarily a subspace.

Note that $V \oplus W$ and $V + W$ are vector spaces, but $V \cup W$ is not. These three spaces are not the same, in fact $V \oplus W$ is never equal to $V + W$ (though there may be a nice function between the two.

Example 5.10. We note some common examples of vector spaces generated by other spaces:

- The vector space generated by V and any of its subspaces W is the original space: $V + W = V$
- The vector space generated by two spans is the span of the union:

$$\text{span}(\{\mathbf{v}_1, \mathbf{v}_2\}) \ + \ \text{span}(\{\mathbf{w}_1, \mathbf{w}_2\}) = \text{span}(\{\mathbf{v}_1, \mathbf{v}_2\} \cup \{\mathbf{w}_1, \mathbf{w}_2\}) = \text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2\})$$

See Exercise 5.6 for more details on why the union of two vector spaces $V \cup W$ is not the same as $+$.

Inquiry 5.11: Let V be a vector space, $X = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq V$, and let S be the span of X.

- Explain why X is always a subset of S, and why V is never a proper subset of S.
- When $V = \mathcal{M}_{2\times 2}$ and $X = \left\{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right\}$, prove that $S \neq V$ by finding an element in $V \setminus S$.
- How big does X have to be for S to be all of V? Is there a limit to how small it can be?

When $S = V$, we say that V is *spanned* by X.

5.2 The column space of a matrix

A big reason we are talking about vector spaces is that the matrix product $A\mathbf{x}$ from the matrix equation $A\mathbf{x} = \mathbf{b}$, over all possibilities $\mathbf{x}$, describes a vector space. This space has a particular name.

Definition 5.12: For an $m \times n$ matrix A, the *column space* of A, denoted $\text{col}(A)$, is the set of all vectors $\mathbf{v} \in \mathbf{R}^m$ that are linear combinations of the columns of A. That is,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \text{col}(A) = \left\{ c_1 \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \;:\; c_i \in \mathbf{R} \right\}.$$

Since every element in $\text{col}(A)$ is a linear combination of vectors, $\text{col}(A)$ is a subspace of $\mathbf{R}^m$.

Example 5.13. Consider $A = \left[\begin{smallmatrix} 3 & -1 & -2 & 4 \\ 0 & 2 & -2 & 1 \end{smallmatrix}\right]$, for which

$$\text{col}(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \;:\; c_i \in \mathbf{R} \right\}.$$

Note that $\left[\begin{smallmatrix} 5 \\ 6 \end{smallmatrix}\right] \in \text{col}(A)$, as

$$\begin{bmatrix} 5 \\ 6 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Inquiry 5.14: Let $A \in \mathcal{M}_{m\times n}$.

- Show that $\left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right] \in \text{col}\left(\left[\begin{smallmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right]\right)$.
- Suppose that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \in \text{col}(A)$. Explain why $A\mathbf{x} = \mathbf{v}$ has a solution. What is it?
- Suppose that $\mathbf{x} = (x_1, \ldots, x_n)$ solves the equation $A\mathbf{x} = \mathbf{b}$. Explain why $\mathbf{b} \in \text{col}(A)$.
- Explain why $0 \in \text{col}(A)$. Hint: What is a solution to $A\mathbf{x} = 0$?

Example 5.15. Consider the following matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

The column space $\text{col}(I)$ is all of $\mathbf{R}^2$, since any vector $(a, b) \in \mathbf{R}^2$ can be described as $a\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + b\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, which is a linear combination of the columns of I. The column space of A is all multiples of the vector $\left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]$, since the second and third rows are multiples of the first row.

5.3 The nullspace of a matrix

Another big reason we are talking about vector spaces is another link to the matrix equation $A\mathbf{x} = \mathbf{b}$, in the special case that $\mathbf{b} = 0$. All the vectors $\mathbf{x}$ satisfying this equation form a vector space. Note that all the vectors satisfying $A\mathbf{x} = \mathbf{b}$ did **not** form a vector space for arbitrary $\mathbf{b}$ - the column space was the space of all vectors $A\mathbf{x}$, not just $\mathbf{x}$.

Definition 5.16: For an $m \times n$ matrix A, the *nullspace* of A is the set

$$\text{null}(A) = \{\mathbf{x} \in \mathbf{R}^n \ : \ A\mathbf{x} = 0\}.$$

The nullspace is a vector space. The nullspace lives inside $\mathbf{R}^n$, but the column space lives in $\mathbf{R}^m$. To find the nullspace of A, we use Gaussian and Gauss–Jordan elimination on A. We may perform row swaps at the beginning or in the middle of elimination, it will not change the result.

Example 5.17. The nullspace of the matrix $A = \left[\begin{smallmatrix}2 & -1\\4 & -2\end{smallmatrix}\right]$ consists of the vectors in $\mathbf{x} \in \mathbf{R}^2$ for which $A\mathbf{x} = 0$. The second row is a multiple of the first (and the second column is a multiple of the first) , so the nullspace is all pairs (x_1, x_2) for which $2x_1 - x_2 = 0$, or $x_1 = x_2/2$. Choosing $x_2 = 1$ (though we could choose any other value) we get $x_1 = 1/2$, so the nullspace is

$$\text{null}\left(\begin{bmatrix}2 & -1\\4 & -2\end{bmatrix}\right) = \left\{\begin{bmatrix}x_2/2\\x_2\end{bmatrix} \ : \ x_2 \in \mathbf{R}\right\} = \text{span}\left(\begin{bmatrix}1/2\\1\end{bmatrix}\right).$$

The choice $(1/2, 1)$ was a *special solution*, but there are many other solutions.

Remark 5.18. Elimination on a matrix does not change its nullspace. We can see this by considering the original equation $A\mathbf{x} = 0$ and the elminiated equation $EA\mathbf{x} = 0$. Since E is an elmentary matrix, it has an inverse, so $A\mathbf{x} = E^{-1}0 = 0$. Hence $\mathbf{x}$ satisfies the first equation iff it satisfies the second equation.

Example 5.19. We describe how to compute the nullspace by way of an example, on $A = \left[\begin{smallmatrix}2 & -2 & 2 & 4 & 8\\1 & 5 & -3 & 0 & 1\\3 & 3 & -1 & -5 & 6\end{smallmatrix}\right]$. We begin with Gaussian elmination to get zeros below the first pivot. The multipliers are given below, and zeros of pivot columns are highlighted:

$$\ell_{21} = \frac{1}{2}, \quad \ell_{31} = \frac{3}{2}: \quad \begin{bmatrix}2 & -2 & 2 & 4 & 8\\0 & 6 & -4 & -2 & -3\\0 & 6 & -4 & -11 & -6\end{bmatrix}.$$

We continue to get a zero below the second pivot:

$$\ell_{32} = 1: \quad \begin{bmatrix}2 & -2 & 2 & 4 & 8\\0 & 6 & -4 & -2 & -3\\0 & 0 & 0 & -9 & -3\end{bmatrix}.$$

The third pivot is -9. Now we move upward and clear the entries above the third pivot:

$$\begin{bmatrix}2 & -2 & 2 & 0 & 20/3\\0 & 6 & -4 & 0 & -7/3\\0 & 0 & 0 & -9 & -3\end{bmatrix}.$$

Next, get a zero above the second pivot:

$$\begin{bmatrix} 2 & 0 & 2/3 & 0 & 53/9 \\ 0 & 6 & -4 & 0 & -7/3 \\ 0 & 0 & 0 & -9 & -3 \end{bmatrix}.$$

Finally, multiply through by the pivot reciprocals to get pivots that are 1:

$$\begin{bmatrix} 1 & 0 & 1/3 & 0 & 53/18 \\ 0 & 1 & -2/3 & 0 & -7/18 \\ 0 & 0 & 0 & 1 & 1/3 \end{bmatrix}.$$

We pause this example for a few comments.

Definition 5.20: The form of A in the example above is called the *reduced row echelon form*, or *RREF*, of A. More specifically:

- columns 1,2,4 are the *pivot columns*,
- columns 3,5 are the *free columns*.

In the equation $A\mathbf{x} = 0$, for $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$, the variables x_1, x_2, x_4 are the *pivot variables* and x_3, x_5 are the *free variables*.

We continue solving for the nullspace null(A) from Example 5.19. It is defined as a linear combination of as many vectors as there are free columns. Each free column gives a nonzero $\mathbf{x}$ that will be in the nullspace, by setting that free variable to 1, all other free variables to 0, and choosing the earlier pivot variables to be the negative entries in those rows:

$$\begin{bmatrix} 1 & 0 & 1/3 & 0 & 53/18 \\ 0 & 1 & -2/3 & 0 & -7/18 \\ 0 & 0 & 0 & 1 & 1/3 \end{bmatrix} \underbrace{\begin{bmatrix} -1/3 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{s}_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 1/3 & 0 & 53/18 \\ 0 & 1 & -2/3 & 0 & -7/18 \\ 0 & 0 & 0 & 1 & 1/3 \end{bmatrix} \underbrace{\begin{bmatrix} -53/18 \\ 7/18 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}}_{\mathbf{s}_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The two vectors $\mathbf{s}_2, \mathbf{s}_2$ are the *special solutions* for the nullspace of A. Hence the nullspace is

$$\text{null}(A) = \{c_1\mathbf{s}_1 + c_2\mathbf{s}_2 \ : \ c_1, c_2 \in \mathbf{R}\} = \left\{ c_1 \begin{bmatrix} -1/3 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -53/18 \\ 7/18 \\ 0 \\ -1/3 \\ 1 \end{bmatrix} \ : \ c_1, c_2 \in \mathbf{R} \right\},$$

so for example, something like

$$\begin{bmatrix} -108 \\ 18 \\ 6 \\ -13 \\ 36 \end{bmatrix} = 6 \begin{bmatrix} -1/3 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 36 \begin{bmatrix} -53/18 \\ 7/18 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}$$

is in the nullspace.

Algorithm 2 (Computing the nullspace): To compute the nullspace of $A \in \mathcal{M}_{m\times n}$, we do Gaussian and Gauss–Jordan elimination so that all columns with pivots have 1's as the only entry.

1. Perform Gauss–Jordan elimination on A to clear all entries below the pivots. The matrix is now A'.
2. Perform Gaussian elimination on A' to clear all above below the pivots. The matrix is now A''.
3. Multiply A'' by diagonal matrices to make all the pivots 1's.
4. Suppose columns $c_1, \dots, c_k$ are pivot columns, and columns $f_1, \dots, f_\ell$ are free columns.
   (a) The nullspace will be a span $(\mathbf{v}_1, \dots, \mathbf{v}_\ell)$ of as many vectors as free columns. The vector $\mathbf{v}_i \in \mathbf{R}^{k+\ell}$ has:
   (b) entry 1 in row f_i and entry 0 in all other rows $f_{j\neq i}$
   (c) entry in row c_j the same, but negative, as the entry in column f_i and row j of A''

Remark 5.21. Note that the pivot columns create an identity matrix in RREF of A, which were highlighted in green and yellow in the main example above. Similarly, the free variable rows in the special solutions create an identity matrix.

Inquiry 5.22: Consider the matrix $A = \begin{bmatrix} 3 & 6 & -1 & 0 & 1 \\ 9 & 18 & -3 & 2 & 0 \\ 0 & 0 & -5 & 1 & 1 \end{bmatrix}$.

- Compute null(A) as the span of vectors.
- Construct a matrix B for which null$(A) = $ col(B).
- Compute col(A) as the span of vectors.
- Do you have to use all the columns of A? That is, are some columns linear combinations of others? Try to use as few columns of A as possible to express col(A) as a span.

5.4 Exercises

Exercise 5.1. Check that the subspace $W \subseteq V$ from in the fourth example in Example 5.2 satisfies the conditions of being a vector space from Definition 5.1.

Exercise 5.2. Let $V = \text{span}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\})$ and $W = \text{span}(\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}\})$. Show that $W \subseteq V$.

Exercise 5.3. Consider the set X of all functions $f\colon \mathbf{R} \to \mathbf{R}$.

1. If addition on X is defined as usual, with $f + g = f(x) + g(x)$, but multiplication is defined as $cf = f(cx)$, show that X is not a vector space.
2. If multiplication is defined as usual, with $cf = cf(x)$, but addition is defined as $f + g = f(g(x))$, but show that X is not a vector space.

Exercise 5.4. Construct the nullspace of $A = \begin{bmatrix} 2 & 9 & 1 & 0 & 0 & 9 \\ 0 & -3 & 0 & 1 & 0 & -3 \\ 8 & -6 & 0 & 0 & 1 & 1 \end{bmatrix}$ as a span of vectors.

Exercise 5.5. Let V be a vector space.

1. Explain why span$(V) = V$ and span$(\{0\}) = \{0\}$.

2. For $V = \mathbf{R}^3$, give an example of $A, B \in \mathcal{M}_{3\times 3}$ with $\operatorname{col}(A) = V$ and $\operatorname{null}(B) = V$. Explain why A and B can not be the same matrix.

Exercise 5.6. Consider the following vector spaces:

$$V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

1. Show that $\mathbf{R}^3$ is a subspace of $V + W$ by describing an arbitrary vector $(x, y, z) \in \mathbf{R}^3$ as a linear combination of the elements of V and W.

2. Show that $V \cup W \neq V + W$ by finding a vector in $V + W$ that is not in $V \cup W$.

Exercise 5.7. Create a matrix with no zero columns that has:

1. size 3×3 and column space the xy-plane (that is, all linear combinations of $(1, 0, 0)$ and $(0, 1, 0)$)

2. size 3×4 and column space the xy-plane

3. size 2×2, column space all of $\mathbf{R}^2$, not a multiple of I_2, and no zero entries. Describe $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ as linear combinations of the columns.

Exercise 5.8. Let I be the 2×2 identity matrix. For each of the following matrices, bring it to RREF and drescribe its nullspace as a span of vectors.

$$A = \begin{bmatrix} I & I \end{bmatrix} \qquad B = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \qquad C = \begin{bmatrix} I & I \\ I & I \end{bmatrix}$$

Lecture 6: Completely solving $A\mathbf{x} = \mathbf{b}$

Chapter 3.3 in Strang's "Linear Algebra"

- Fact 1: The complete solution to $A\mathbf{x} = \mathbf{b}$ consists of the particular solution and linear combinations of the special solutions.
- Fact 2: The rank of a matrix is the number of pivots. It can not be larger than the number of rows or columns.

- Skill 1: Construct the complete solution to any matrix $A \in \mathcal{M}_{m\times n}$.
- Skill 2: Identify the row rank, column rank, rank of a matrix.

Previously we saw how to solve $A\mathbf{x} = 0$, by doing elimination until we get an upper triangular matrix $R\mathbf{x} = 0$, whose solutions $\mathbf{x}$ are the same solutions that solve the first equation. In this lecture we generalize to finding soluetions to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is not necessarily the zero vector.

6.1 Rank and the particular solution

We begin with the example from the previous lecture,

$$A = \begin{bmatrix} 2 & -2 & 2 & 4 & 8 \\ 1 & 5 & -3 & 0 & 1 \\ 3 & 3 & -1 & -5 & 6 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 1/3 & 0 & 53/18 \\ 0 & 1 & -2/3 & 0 & -7/18 \\ 0 & 0 & 0 & 1 & 1/3 \end{bmatrix}, \qquad EA = R$$

for some product of elimination matrices E. The columns 1,2,4 are the *pivot columns* and the columns 3,5 are the *free columns* (this is true for both R and A). It is immediate that columns 1,2,4 of R can not be written one as a linear combination of the others - that is, these three columns are *linearly independent*. Again, this is true for both R and A.

Definition 6.1: The *rank* of a matrix $A \in \mathcal{M}_{m\times n}$ is denoted $\operatorname{rank}(A)$, and is equivalently

- the number of pivots of A, or
- the number of columns in A that are not linear combinations of other rows.

If $\operatorname{rank}(A) = \min(m, n)$, then A is said to have *full rank*.

Reducing the matrix A to RREF reveals which columns are combinations of others. Since only row operations were performed, any linear relationships among the columns are preserved.

Example 6.2. When a matrix has rank 1, all the columns are multiples of the first one. For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank one, and its column space is all the multiples of $(1, 1, 1)$. To find its nullspace, we look at its RREF, which has special solutions

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

hence the nullspace of A is the span of $\left[\begin{smallmatrix} -2 \\ 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -3 \\ 0 \\ 1 \end{smallmatrix}\right]$.

Remark 6.3. A rank 1 square $n \times n$ matrix may be expressed as a product of a $n \times 1$ vector with a $1 \times n$ vector, since all the columns are multiples of the first column. For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \mathbf{v}\mathbf{w}^T.$$

Example 6.4. The identity matrix I has full rank. The zero matrix 0 has rank 0.

Inquiry 6.5: Consider the matrix $A = \left[\begin{smallmatrix} a & b & c \\ b & c & b \\ c & a & a \end{smallmatrix}\right]$.

- Find values of a, b, c for which A has rank 0,1, 2, 3.
- Suppose another column was added at the end of A to make $\left[\begin{smallmatrix} a & b & c & 0 \\ b & c & b & 0 \\ c & a & a & 0 \end{smallmatrix}\right]$. Explain why your answers to the first part above would not change using this matrix.

Definition 6.6: The number of special solutions to $A\mathbf{x} = 0$ is called the *nullity* of A.

The nullity is the number of free columns of A, and the smallest number of vectors that can be used to define null(A) as a span. If $A \in \mathcal{M}_{n\times n}$ is square, then using the fact that the rank is the number of pivot columns, we immediately get that

$$\text{rank}(A) + \text{nullity}(A) = n, \tag{2}$$

a very powerful equation, more of which we will see later. This is called the *rank-nullity theorem*.

Example 6.7. Recall Example 5.19 from Lecture 5. Suppose that instead of $A\mathbf{x} = 0$, we considered $A\mathbf{x} = \mathbf{b}$, which, after elimination, would become $R\mathbf{x} = \mathbf{d} = [d_1\ d_2\ d_3]^T$. The vector $\mathbf{x} = 0$ is not a solution anymore, but we can find a quick solution by setting the variables corresponding to the free columns equal to 0:

$$\begin{bmatrix} 1 & 0 & 1/3 & 0 & 53/18 \\ 0 & 1 & -2/3 & 0 & -7/18 \\ 0 & 0 & 0 & 1 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad \text{is solved by} \qquad \begin{aligned} x_1 &= d_1, \\ x_2 &= d_2, \\ x_3 &= d_3. \end{aligned}$$

The vector $(d_1, d_2, 0, d_3, 0)$ is called a *particular solution* to $A\mathbf{x} = \mathbf{b}$. This particular solution solves not only $R\mathbf{x} = \mathbf{d}$, but also $A\mathbf{x} = \mathbf{b}$, because if $A = ER$, for some elimination matrix E, then $\mathbf{d} = E\mathbf{b}$.

Remark 6.8. What we have done so far can be summarized as follows:

- The special solutions $\mathbf{x} = \mathbf{s}_1, \mathbf{s}_2$ solve $A\mathbf{x} = 0$
- The particular solution $\mathbf{x} = \mathbf{p}$ solves $A\mathbf{x} = \mathbf{b}$

Finally, the *complete solution* to the system $A\mathbf{x} = \mathbf{b}$ is the sum of the particular and special solutions. That is, $\mathbf{x} = \mathbf{p} + c_1\mathbf{s}_1 + c_2\mathbf{s}_2$ solves the system, for any $c_1, c_2 \in \mathbf{R}$, because

$$A(\mathbf{p} + c_1\mathbf{s}_1 + c_2\mathbf{s}_2) = A\mathbf{p} + c_1 A\mathbf{s}_1 + c_2 A\mathbf{s}_2 = \mathbf{b} + c_1 \cdot 0 + c_2 \cdot 0 = \mathbf{b}.$$

Algorithm 3 (Finding the complete solution): Consider the matrix equation $A\mathbf{x} = \mathbf{b}$.

1. Compute the nullspace of $[A \mid \mathbf{b}]$. That is, find the special solutions $\mathbf{s}_1, \ldots, \mathbf{s}_k$ by doing elimination on the augmented matrix $[A \mid \mathbf{b}]$.

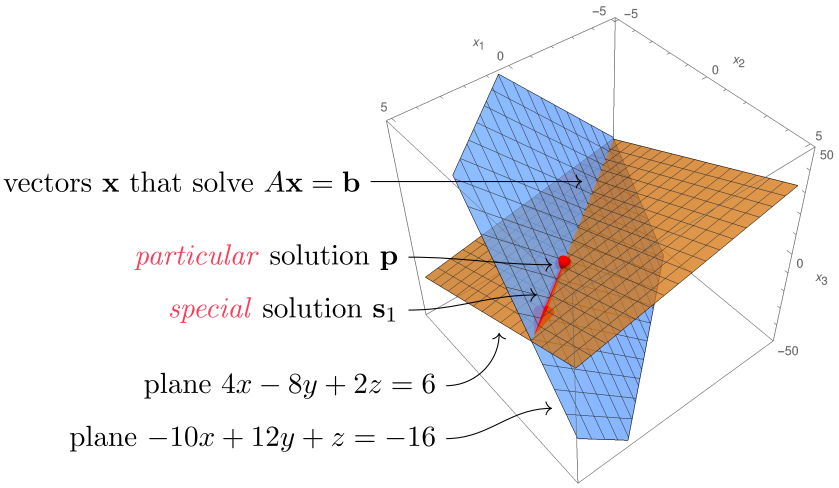
2. Elimination on $[A \mid \mathbf{b}]$ produces the matrix $[R \mid \mathbf{d}]$. Construct the particular solution $\mathbf{p}$ from $\mathbf{d}$ as in Example 6.7.

3. The complete solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{p} + c_1\mathbf{s}_1 + \cdots + c_k\mathbf{s}_k$, for all $c_i \in \mathbf{R}$.

Example 6.9. Consider the matrix equation $A\mathbf{x} = \mathbf{b}$, in the form

$$\underbrace{\begin{bmatrix} 4 & -8 & 2 \\ -10 & 12 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 6 \\ -16 \end{bmatrix}}_{\mathbf{b}} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/4 \\ 1/8 \end{bmatrix}.$$

The complete solution to this equation is $\mathbf{x} = \begin{bmatrix} 7/4 \\ 1/8 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 3/4 \\ 1 \end{bmatrix}$, for any $c_1 \in \mathbf{R}$.



This equation represents two planes intersecting in space, as in the picture above. The particular solution is a point on the line of intersection and the special solution is a vector in the direction of the line. The line of intersection is all the vectors $\mathbf{x}$ that make $A\mathbf{x} = \mathbf{b}$ true. In other words, it the nullspace shifted by the vector $\mathbf{p}$, hence it is an *affine space*.

In the example above, the two planes are defined by the initial equation. After row reduction, we have two different planes which still have same intersection. Compare this with the 2-dimensional row picture presented in Example 3.5.

Inquiry 6.10: Consider the nullspace null(A) from Example 6.9.

- Write the nullspace null(A) as the span of a single vector.

- Let S be the set of all solutions $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{b}$. Explain why S is not a vector space in the same way that null(A) is a vector space.

- Using Inquiry 5.3, explain why S still has some vector space structure. This type of space is an ________ space.

6.2 Different types of complete solutions

Now we consider the implications for the complete solution given the rank of the matrix. Recall from Definition 6.1 that $A \in \mathcal{M}_{m\times n}$ has *full rank* if it has A has $\min(m,n)$ pivots.

Definition 6.11: Let $A \in \mathcal{M}_{m\times n}$.

- If each row of A has a pivot (so A has m pivots), then A has *full row rank*.
- If each column of A has a pivot (so A has n pivots), then A has *full column rank*.

Example 6.12. Consider the following types of common situations for rank, for $A \in \mathcal{M}_{m\times n}$. If A has more rows than columns (so $m > n$) and has full column rank, then in row reduced echelon form A looks like the block matrix $\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, where I is of size $n \times n$ and the zero matrix 0 has size $(m-n)\times n$. Then:

- all columns of A are pivot columns,
- there are no free variables, so there are no special solutions,
- the nullspace contains only the zero vector $\text{null}(A) = \{0\}$,
- if $A\mathbf{x} = \mathbf{b}$ has a solution, there is one unique solution.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Analogously, If A has more columns than rows (so $n > m$) and has full row rank, then in row reduced echelon form A looks like the block matrix $[I\ 0]$, where I is of size $m \times m$ and the zero matrix 0 has size $m \times (n-m)$. Then:

- all rows of A have pivots, so there are no zero rows,
- there are $n - m$ special solutions,
- the column space is all of $\mathbf{R}^m$,
- $A\mathbf{x} = \mathbf{b}$ has a solution for any vector $\mathbf{b}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Example 6.13. The equation $A\mathbf{x} = \mathbf{b}$ with A an $m \times 3$ matrix with full column rank represents m planes intersecting in 3-dimensional space $\mathbf{R}^3$. If the planes all intersect in one point, there is a solution to this equation.

- For $1 \leqslant m < 3$ and m randomly chosen planes, it is impossible for them to intersect in one point.
- For $m = 3$ and three randomly chosen planes, they will almost always intersect in one point.
- For $m > 3$ and m randomly chosen planes, they will almost never intersect in one point.

The general theory behind these claims has to do with *general position* of points in $\mathbf{R}^3$, and the fact that three points are necessary to define a plane.

Inquiry 6.14: Let $A \in \mathcal{M}_{m\times n}$ and $\mathbf{b} \in \mathbf{R}^m$. Find an example of A and $\mathbf{b}$ so that $A\mathbf{x} = \mathbf{b}$ has:

- exactly one solution, with $m = n = 2$;
- no solutions, with $m = n = 2$;
- exactly one solution, with $m = 3$, $n = 2$;
- no solutions, with $m = 3$, $n = 2$;
- infinitely many solutions, with $m = 2$, $n = 3$.
- Explain why $A\mathbf{x} = \mathbf{b}$ can not have exactly one solution if $n > m$. That is, show that if it has one solution, it has infinitely many.

Remark 6.15. We can summarize every matrix $A \in \mathcal{M}_{m\times n}$ as one of the following four situations.

- $\text{rank}(A) = m$, $\text{rank}(A) = n$: Then A is square and invertible, and $A\mathbf{x} = \mathbf{b}$ has exactly 1 solution.
- $\text{rank}(A) = m$, $\text{rank}(A) < n$: Then A is wider than it is taller, and $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
- $\text{rank}(A) < m$, $\text{rank}(A) = n$: Then A is taller than it is wider, and $A\mathbf{x} = \mathbf{b}$ has 0 or 1 solution, depending on what the bottom row(s) of $[A \mid \mathbf{b}]$ look like in RREF.
- $\text{rank}(A) < m$ and $\text{rank}(A) < n$: Then A can have any shape, but it is not full rank, and $A\mathbf{x} = \mathbf{b}$ has either 0 or infintely many solutions.

6.3 Exercises

Exercise 6.1. Consider the two vectors $\mathbf{v} = [a\ a\ a\ a]^T$ and $\mathbf{w} = [1\ 1\ 1\ 1]^T$. What will be the rank of the 4×4 matrix $\mathbf{v}\mathbf{w}^T$? Your answer should depend on a.

Exercise 6.2. Find the complete solution to $A\mathbf{x} = \mathbf{b}$, for

$$A = \begin{bmatrix} 3 & 0 & -9 & -3 & 0 \\ 6 & 0 & -21 & 0 & 2 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}.$$

Exercise 6.3. Suppose you know that the solution to a matrix equation $A\mathbf{x} = \mathbf{b}$, where $A \in \mathcal{M}_{2\times 3}$, is the vector

$$\mathbf{x} = \begin{bmatrix} 7 \\ 4 \\ -2 \end{bmatrix} + c \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix},$$

for any $c \in \mathbf{R}$.

1. Construct one possible matrix A and vector $\mathbf{b}$ for which this could be the solution.
2. Do the same as above, but make it so that A has no zero entries.

Exercise 6.4. For the following matrices A, B, find the ranks of A^TA, AA^T, B^TA, BB^T:

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 3 \\ 9 & 0 \\ 7 & 0 \\ -3 & 1 \end{bmatrix}.$$

Lecture 7: Independence, basis, dimension

Chapter 3.4 in Strang's "Linear Algebra"

- Fact 1: A basis of a vector space V is a smalest possible set of vectors that spans V
- Fact 2: Bases of V are not unique. The size of a basis is unique - it is the dimension of V.

- Skill 1: Identify linearly independent subsets in a given set of vectors.
- Skill 2: Extend a set of vectors of V to a basis of V.
- Skill 3: Express the same vector in different bases.

We have now arrived at the next big theme of this course: *dimension*.

7.1 Linear independence

Recall that the rank of a matrix A was the number of pivots A had, or the number of columns of A that are not linear combinations of the other columns. A more precise way to say the second approach is with *linear independence*.

Definition 7.1: Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq \mathbf{R}^m$ be the columns of a matrix $A \in \mathcal{M}_{m\times n}$. These vectors are *linearly independent* if, equivalently,

- the only solution to $A\mathbf{x} = 0$ is $\mathbf{x} = 0$, or
- $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = 0$ implies $x_i = 0$ for all i, or
- the nullspace of A is only the zero vector, that is, $\text{null}(A) = \{0\}$.

If a set of vectors is not linearly independent, then the set is *linearly dependent*.

Every set of vectors is either linearly independent or linearly dependent, there is no in-between. We often say "the vectors are linearly independent" instead of "the set of vectors is linearly independet", but both are correct uses of the term.

Example 7.2. Slight changes in the matrix entries can lead to big differences:

- The vectors $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$ are linearly dependent, because $\left[\begin{smallmatrix}1&2\\1&2\end{smallmatrix}\right]\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$.
- The vectors $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}2\\2.001\end{smallmatrix}\right]$ are linearly independent, because attempting to solve $A\mathbf{x} = 0$ will lead to $\mathbf{x} = 0$.

Inquiry 7.3: Recall the three different ways to express linear independence.

- Pick any 3 vectors in $\mathbf{R}^2$. Explain why they must be linearly dependent. *Hint: put them as columns in a matrix and say something about its nullspace.*
- Does the above work for any $4, 5, \ldots$ vectors in $\mathbf{R}^2$? What about any 2 vectors?
- Try to generalize the above points into a statement like: "Any set of more than ____ vectors in ____ will be linearly ____."

Recall the *span* of a collection of vectors from Definition 5.6 and Inquiry 5.11, and the columns of a matrix *spanning* its column space, as well as the vectors from special solutions *spanning* the nullspace.

Definition 7.4: Let V be a vector space and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subseteq V$. If $V = \text{span}(S)$,

- S is called a *spanning set* of V, and
- S is called a *minimal spanning set* of V if for every other spanning set S' of V, the size of S is less than or equal to the size of S'.

Example 7.5. Minimal spanning sets are common.

- The vector space $\mathbf{R}^3$ has a minimal spanning set in $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$.
- The pivot columns of a matrix form a minimal spanning set for its column space.

The idea of a minimal spanning set from Definition 7.4 can be made more precise with the idea of linear independence from Definition 7.1.

Definition 7.6: Let V be a vector space and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subseteq V$. The set S is a *basis* for V if, equivalently,

- S is a minimal spanning set for V, or
- S spans V, and S is linearly independent, or
- every $\mathbf{v} \in V$ can be written uniquely as $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$, for $a_i \in \mathbf{R}$.

Example 7.7. The *standard basis* for $\mathbf{R}^3$ consists of the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$. In general, the *standard basis* for $\mathbf{R}^n$ consists of the n column vectors of the $n \times n$ identity matrix, and they are often denoted $\mathbf{e}_1, \dots, \mathbf{e}_n$:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} | & | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \cdots & \mathbf{e}_n \\ | & | & | & & | \end{bmatrix}.$$

The standard basis is not the only basis for $\mathbf{R}^n$, and the columns of every full rank $n \times n$ matrix will give a basis for $\mathbf{R}^n$.

Example 7.8. Let $A \in \mathcal{M}_{m \times n}$.

- A basis for the nullspace null(A) is the set of special solutions to $A\mathbf{x} = 0$.
- A basis for the column space col(A) is the pivot columns of A - this is not necessarily all the columns of A.

Algorithm 4 (Find linearly independent vectors in a set): Given a set of vectors in $\mathbf{R}^n$, we can find which of them are linearly independent by either:

- making them columns of a matrix, doing elimination (with row swaps), and taking the positions of the pivot columns, or,
- making them rows of a matrix, doing elimination (without row swaps), and taking the positions of pivots rows.

Inquiry 7.9: Consider the vectors $\mathbf{u}_1 = \left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right], \mathbf{u}_1 = \left[\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right], \mathbf{u}_3 = \left[\begin{smallmatrix}-1\\1\\1\end{smallmatrix}\right], \mathbf{v} = \left[\begin{smallmatrix}1\\2\\4\end{smallmatrix}\right]$ in $\mathbf{R}^3$.

- Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly independent set.
- Express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Suppose that $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$ for some real numbers a_1, a_2, a_3 (not necessarily the ones you got above).

- Without using the definition of a basis (Definition 7.6), but instead using the the first definition of linear independence (Definition 7.1), explain why the a_i must be the same numbers you got above.

Hint: subtract your way to write $\boldsymbol{v}$ from the $\boldsymbol{v} = a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + a_3\boldsymbol{u}_3$ way to write $\boldsymbol{v}$.

Example 7.10. Consider the following three vectors in $\mathbf{R}^4$:

$$\mathbf{u} = \begin{bmatrix}3\\2\\7\\1\end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix}1\\-1\\2\\3\end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix}5\\5\\12\\5\end{bmatrix}.$$

As columns of a matrix, we quickly eliminate entries below the diagonal to identify the the first two as pivot columns and the last as a free column:

$$\begin{bmatrix}3 & 1 & 5\\2 & -1 & 5\\7 & 2 & 12\\1 & 3 & 5\end{bmatrix} \xrightarrow{G.E.} \begin{bmatrix}3 & 1 & 5\\0 & -5/3 & 13/3\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}.$$

So $\mathbf{u}, \mathbf{v}$ are indepdent, and $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Alternatively, we can make them rows of a matrix, and the perform Gaussian elimination (without row swaps). That will give us zero rows, which will correspond to linearly dependent vectors:

$$\begin{bmatrix}3 & 2 & 7 & 1\\1 & -1 & 2 & 3\\5 & 5 & 12 & 5\end{bmatrix} \xrightarrow{G.E.} \begin{bmatrix}3 & 2 & 7 & 1\\0 & -5/3 & -1/3 & 8/3\\0 & 0 & 0 & 0\end{bmatrix}.$$

As in the first approach, we get that $\mathbf{w}$ depends on $\mathbf{u}$ and $\mathbf{v}$. Hence $\{\mathbf{u}, \mathbf{v}\}$ is a basis for the vector space $V = \mathrm{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

7.2 Dimension and extending to a basis

The key idea from the first part of this lecture is that rhe word *basis* is another name for *minimal spanning set*. It is often difficult to consider all possible spanning sets, so we use the word *basis* much more often. Keep in mind three important things:

- bases are not unique,
- every basis of a vector space must have the same number of vectors, and
- every vector space has a basis.

The last conclusion is based on a fundamental (and unproven!) cornerstone of mathematics called the *axiom of choice*. A special case is investigated in Inquiry 7.11.

Inquiry 7.11: Consider the vector space $\mathbf{R}^4$ with its four standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_4$, as given in Example 7.7. Consider $\mathbf{u} = \left[\begin{smallmatrix} 1 & 2 & 3 & 4 \end{smallmatrix}\right]^T, \mathbf{v} = \left[\begin{smallmatrix} 0 & 1 & -1 & 2 \end{smallmatrix}\right]^T \in \mathbf{R}^4$

- Explain why $\{\mathbf{u}, \mathbf{v}\}$ cannot be a basis of $\mathbf{R}^4$. Is it linearly (in)dependent?
- Explain why $\{\mathbf{u}, \mathbf{v}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ cannot be a basis of $\mathbf{R}^4$. Is it a linearly (in)dependent?
- Find a linearly independent subset of $\{\mathbf{u}, \mathbf{v}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ that contains $\mathbf{u}$ and $\mathbf{v}$.

The last point is called *extending* a set to a basis.

Remark 7.12. As mentioned in Definition 7.6, given a basis for a vector space V, every vector in V can be expressed uniquely as a linear combination of vectors of that basis. For some vector space it is very obvious:

$$\begin{bmatrix} 4 \\ -2 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 8 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

However, if we have a different basis, how can we figure out what the linear combination is in the other basis? This is where the *change of basis matrix* appears. Suppose that B and B' are bases for V, with

$$B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}, \qquad B' = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}, \qquad \mathbf{v} \in V.$$

The coefficients for expressing $\mathbf{v}$ in the basis B are in the solution vector $\mathbf{x}$ to $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_k]\mathbf{x} = \mathbf{v}$. Similarly, the coefficients for expressing $\mathbf{v}$ in the basis B' are in the solution vector $\mathbf{y}$ to $[\mathbf{w}_1 \ \cdots \ \mathbf{w}_k]\mathbf{y} = \mathbf{v}$. These two vectors are related by the equation

$$\mathbf{y} = \underbrace{\begin{bmatrix} | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_k \\ | & & | \end{bmatrix}}_{\text{change of basis matrix}} \mathbf{x}, \qquad \text{where} \qquad \mathbf{v}_i = \begin{bmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_k \\ | & & | \end{bmatrix} \mathbf{a}_i.$$

Example 7.13. Consider the two bases $B = \left\{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]\right\}$ and $B' = \left\{\left[\begin{smallmatrix} 3 \\ 7 \end{smallmatrix}\right], \left[\begin{smallmatrix} -2 \\ -4 \end{smallmatrix}\right]\right\}$ of $\mathbf{R}^2$. To construct the change of basis matrix from B to B', we need to express every vector of B as a linear combination of vectors in B'. We do this by sight:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 3 \\ 7 \end{bmatrix} - 2 \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -3 \begin{bmatrix} 3 \\ 7 \end{bmatrix} - 5 \begin{bmatrix} -2 \\ -4 \end{bmatrix},$$

Hence the change of basis matrix from vectors in the basis B to vectors in the basis B' is

$$A = \begin{bmatrix} -1 & -3 \\ -2 & -5 \end{bmatrix}.$$

For example, taking the vector

$$\begin{bmatrix} -1 \\ -13 \end{bmatrix} = -7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

in the basis B, we have the coefficient vector $\mathbf{x} = \left[\begin{smallmatrix} -7 \\ 6 \end{smallmatrix}\right]$. The coefficient vector $\mathbf{y}$ in the basis B' is given by computing

$$\mathbf{y} = \begin{bmatrix} -1 & -3 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} -7 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 - 18 \\ 14 - 30 \end{bmatrix} = \begin{bmatrix} -11 \\ -16 \end{bmatrix}, \qquad \text{meaning} \qquad \begin{bmatrix} -1 \\ -13 \end{bmatrix} = -11 \begin{bmatrix} 3 \\ 7 \end{bmatrix} - 16 \begin{bmatrix} -2 \\ -4 \end{bmatrix}.$$

Note that the inverse of A will take us back to coefficients in the basis B.

Definition 7.14: Let V be a vector space. The *dimension* of V is the number of vectors in any basis of V. It is denoted $\dim(V)$.

This assumes the previously mentioned very important point: all vector spaces have a basis, which needs the axiom of choice.

Example 7.15. We have already seen dimension, but under different names.

- The dimension of $\mathbf{R}^n$ is n.
- The dimension of the column space of A is the rank of A.
- The dimension of the nullspace is the *nullity* of A.

Recall the definition of $U \oplus V$ and $U + V$ from Definiton 5.9. There we saw that if $U = \text{span}(B)$ and $V = \text{span}(B')$, then $U + V = \text{span}(B \cup B')$. A similar statement holds for dimension.

Remark 7.16. Let V be a vector space with subspaces U, W.

- The intersection $U \cap W$ is a subspace of V
- The sum $+$ of vector spaces satisfies $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$
- The sum $\oplus$ of vector spaces satisfies $\dim(U \oplus W) = \dim(U) + \dim(W)$

The third statement does not need that U, W be subspaces of the same space. Statements like this do not exist for the union of vector spaces spaces, because that is not necessarily a vector space.

Remark 7.17. Let V be a vector space and $U \subseteq V$. If $\dim(U) = \dim(V)$, then $U = V$. This follows by taking the basis $\mathbf{u}_1, \ldots, u_n$ of U, and asking if there are any vectors in V which cannot be expressed as linear combinations of the $\mathbf{u}_i$. If no, then the spaces are the same. If there exists some $\mathbf{v}$, then $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v}$ is a linearly independent set of $n + 1$ vectors in V, which is impossible.

Definition 7.18: Let V be a vector space with $\dim(V) = n$, and $U \subseteq V$ a subspace of dimension $\dim(U) = k$. The *codimension* of U in V is $\text{codim}(U) = n - k$.

For example, lines are codimension 1 in $\mathbf{R}^2$, but codimension 2 in $\mathbf{R}^3$. The set of points in $\mathbf{R}^n$ that satisfy one linear equation (that goes through the origin) is codimension 1.

Example 7.19. The space of $n \times n$ matrices has dimension n^2. It has as a subspace the space of $n \times n$ upper triangular matrices, which has dimension $n(n+1)/2$. For $n = 2$, a basis for each of these spaces is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

in the first case, and

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

in the second case.

7.3 Exercises

Exercise 7.1. Find all sets of size 3 from the vectors below that are linearly independent:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

Exercise 7.2. For a 2×2 matrix, linear independence on the columns only depends on if one column is a multiple of the other.

$\bowtie$ (a) Generate 10 000 random 2×2 matrices, with real number entries in the range $[-5, 5]$. How many have column space dimension 1?

$\bowtie$ (b) Repeat the same as in part (a), but use integer entries in the range $[-5, 5]$. How many have column space dimension 1? **Bonus:** How many would you expect to have dimension 1?

Exercise 7.3. Consider the basis B for $\mathbf{R}^3$ and a vector $\mathbf{v}$,

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} \right\}, \qquad \mathbf{v} = \begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix}.$$

Express $\mathbf{v}$ in terms of B.

Exercise 7.4. Consider the plane $P = \{(x, y, z) \in \mathbf{R}^3 \ : \ 2x - 4y - 5z = 0\}$, which is a subspace of $\mathbf{R}^3$. What is its basis?

Exercise 7.5. Find the change of basis matrix from $\left\{ \left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right] \right\}$ to $\left\{ \left[\begin{smallmatrix} -2 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 5 \end{smallmatrix}\right] \right\}$.

Exercise 7.6. Prove the claims from Remark 7.16.

Exercise 7.7. This question is about vector spaces of matrices, where matrix addition and scalar multiplication are defined as usual.

1. Give a basis for the space of diagonal 3×3 matrices and a basis for the space of skew-symmetric 3×3 matrices.

2. For $n \in \mathbf{N}$, what is the dimension of the space of $n \times n$ diagonal matrices and what is the dimension of the space of $n \times n$ skew-symmetric matrices?

3. Show by example that the set of all invertible 2×2 matrices does not form a vector space. Show that all linear combinations of invertible 2×2 matrices describe the set $\mathcal{M}_{2\times 2}$.
Hint: Construct the basis matrices of $\mathcal{M}_{2\times 2}$ as linear combinatons of invertible matrices.

Lecture 8: The four fundamental subspaces associated to a matrix

Chapter 3.5 in Strang's "Linear Algebra"

- Fact 1: A line in $\mathbf{R}^2$ is given by one equation, in $\mathbf{R}^3$ by two equations.
- Fact 2: Every matrix with m rows splits up $\mathbf{R}^m$ into the column space and the left nullspace.
- Fact 3: Every matrix with n columns splits up $\mathbf{R}^n$ into the row space and the nullspace.

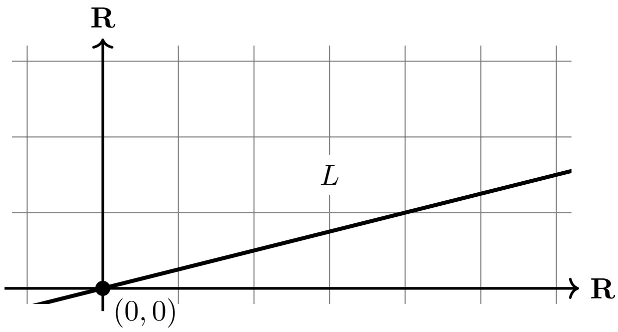
- Skill 1: Find the intersection of two planes.
- Skill 2: Describe a hyperplane as a span of vectors.

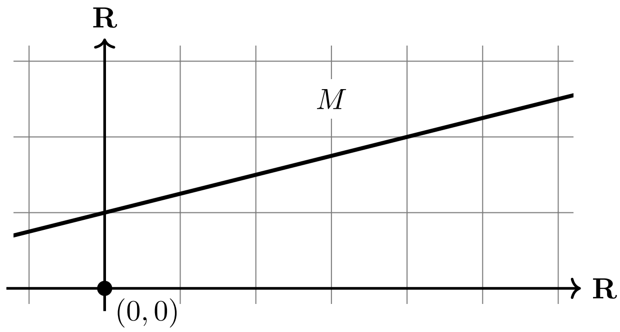
With this lecture we take the column space and nullspace to the transpose matrix, and describe strong relationships among these spaces.

8.1 Lines, planes, and hyperplanes

Since we will be discussing spaces and their relationships with each other in this lecture, we begin with a comparison relating two similar lines.

Example 8.1. Consider the two lines $L, M \subseteq \mathbf{R}^2$ given below.





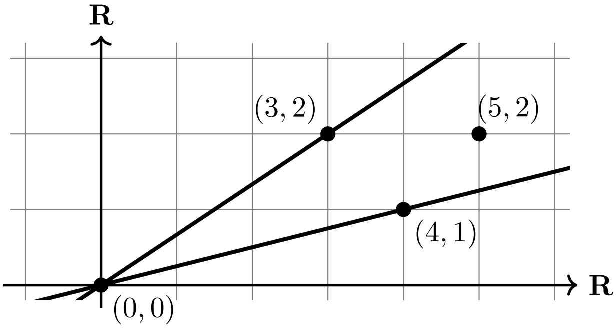
Each of these lines can be considered in similar ways. Each is:

- the line $y = x/4$
- the pairs (x, y) that satisfy $\frac{1}{4}x - y = 0$
- the nullspace of $\begin{bmatrix} 1 & -4 \end{bmatrix}$
- the set of vectors $\mathbf{x}$ for which $\begin{bmatrix} 1 & -4 \end{bmatrix}\mathbf{x} = 0$
- a vector subspace of $\mathbf{R}^2$
- a vector space of dimension 1

- the line $y = x/4 + 1$
- the pairs (x, y) that satisfy $\frac{1}{4}x - y = -1$
- not the nullspace of any matrix
- the set of vectors $\mathbf{x}$ for which $\begin{bmatrix} 1 & -4 \end{bmatrix}\mathbf{x} = -4$
- not a vector subspace of $\mathbf{R}^2$
- an affine space of dimension 1

The line M can be considered as a vector space, using a different addition and multiplication than in $\mathbf{R}^2$. This is the same *affine space* structure seen before in Inquiry 5.3 and Example 6.9.

Note that the vectors in L do not span all of $\mathbf{R}^2$, but if we add another line at a different angle than L, also going through the origin, vectors from both lines together will span $\mathbf{R}^2$.

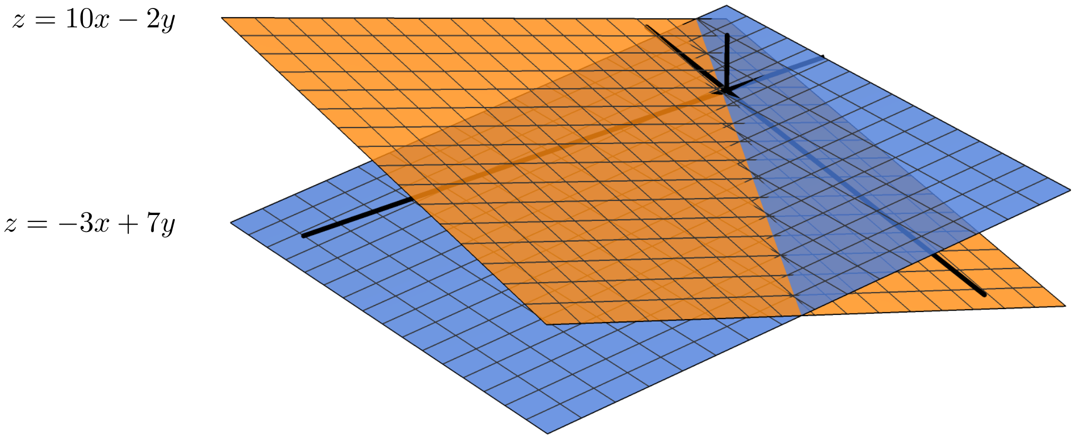


one line: neither of the two lines go through $\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]$

two lines: there is a unique solution to $\left[\begin{smallmatrix}4&3\\1&2\end{smallmatrix}\right]\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]=\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]$

The reason for going through all the different perspectives in Example 8.1 was to connect visual with algebraic intution. We go one step further, into the third dimension, with the following example.

Example 8.2. Consider the two planes in $\mathbf{R}^3$ given below. Note that their intersection is a line.



Both planes go through the origin $(0, 0, 0)$. To find the vector along the line of intersection, we need both equations to be satified at the same time. That is, we want to solve the matrix equation

$$\begin{bmatrix}10 & -2 & 1\\ -3 & 7 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}. \qquad \text{Notice } \begin{bmatrix}10 & -2 & 1\\ -3 & 7 & 1\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 9/64\\ 0 & 1 & 13/64\end{bmatrix},$$

so the nullspace of the matrix on the left is the span of the single vector $\left[\begin{smallmatrix}-9 & -13 & 64\end{smallmatrix}\right]^T$. Unlike in $\mathbf{R}^2$, a line in R^3 can not be described by a single equation. We either use two equations (of the two planes), or a single vector. Finally, we observe that to describe a plane as a span of vectors, we also use the nullspace:

$$\operatorname{null}\left(\begin{bmatrix}10 & -2 & 1\end{bmatrix}\right) = \operatorname{null}\left(\begin{bmatrix}1 & -1/5 & 1/10\end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix}1/5\\1\\0\end{bmatrix}, \begin{bmatrix}-1/10\\0\\1\end{bmatrix}\right).$$

These are the two vectors whose span is the plane $z = 10x - 2y$.

Definition 8.3: A *hyperplane* in $\mathbf{R}^n$ is the set of points that satisfies a single equation $a_1x_1 + \cdots + a_nx_n = 0$.

- For $n = 1$, a hyperplane in $\mathbf{R}^1$ is a *point*.
- For $n = 2$, a hyperplane in $\mathbf{R}^2$ is a *line*.
- For $n = 3$, a hyperplane in $\mathbf{R}^3$ is a *plane*.

A hyperplane is an $(n-1)$-dimensional (or codimension 1) subspace of $\mathbf{R}^n$.

Inquiry 8.4: Consider the vector $\mathbf{v} = [\begin{smallmatrix}1 & 2 & -3\end{smallmatrix}]^T \in \mathbf{R}^3$.

- Construct a 2×3 matrix A so that $\mathbf{v}$ is in the nullspace of A. Explain why $\mathbf{v}$ is in the left nullspace of A^T.
- Find two planes in $\mathbf{R}^3$ so that their intersection is $\mathbf{v}$.
- Construct a 3×3 matrix B so that $\text{col}(B) = \text{span}(\mathbf{v})$. What is the nullspace of B?

8.2 The four fundamental subspaces

Let $A \in \mathcal{M}_{m\times n}$, and let $R \in \mathcal{M}_{m\times n}$ be the result of applying Guassian and Gauss–Jordan elimination to A. We have seen two related vector spaces:

- the *column space* $\text{col}(A) \neq \text{col}(R)$, which is the span of the columns
- the *nullspace* $\text{null}(A) = \text{null}(R)$, which is the span of the (special) solutions to $A\mathbf{x} = 0$ or $R\mathbf{x} = 0$

We now introduce two other spaces, which are related to the above two by the *transpose* of A.

Definition 8.5: Let $A \in \mathcal{M}_{m\times n}$.

- The *row space*, denoted $\text{row}(A)$, is the span of the rows of A.
- The *left nullspace* is the span of the solutions to $\mathbf{x}^TA = 0$.

Together these four vector spaces are the *four fundamental subspaces*.

Remark 8.6. The left nullspace has no special way to write it. Observing that $(\mathbf{x}^TA)^T = A^T\mathbf{x}$, we see that the left nullspace of A is the vector space $\text{null}(A^T)$. With this, we see several other relations among the four fundamental spaces:

$$\text{row}(A) = \text{col}(A^T), \qquad \text{row}(A^T) = \text{col}(A), \qquad \text{null}(A) = \begin{pmatrix}\text{left null-}\\ \text{space of } A^T\end{pmatrix}, \qquad \text{null}(A^T) = \begin{pmatrix}\text{left null-}\\ \text{space of } A\end{pmatrix}.$$

Remark 8.7. The previous remark makes it clear that the row space and left nullspace are vector spaces. Below we put together all the relationships among these four subspaces, for $A \in \mathcal{M}_{m\times n}$.

1. *subspace* relations:
   - $\text{col}(A) \subseteq \mathbf{R}^m$ and $\text{null}(A^T) \subseteq \mathbf{R}^m$ are subspaces
   - $\text{col}(A^T) \subseteq \mathbf{R}^n$ and $\text{null}(A) \subseteq \mathbf{R}^n$ are subspaces
2. *dimension* relations:
   - $\dim(\text{col}(A)) = \dim(\text{col}(A^T)) = \text{rank}(A) = \text{rank}(A^T)$

- $\dim(\text{col}(A)) + \dim(\text{null}(A^T)) = m$
- $\dim(\text{col}(A^T)) + \dim(\text{null}(A)) = n$

The last statement is called the *rank-nullity theorem*, which we already saw just after Definition 6.6. We now look at more relations among these vector spaces.

Inquiry 8.8: Consider the matrix $A = \left[\begin{smallmatrix} 2 & 0 & -1 \\ 1 & -2 & 1 \end{smallmatrix}\right]$.

- Describe the column space of A as the span of two nonzero vectors.
- Suppose your answer to the above was $\text{col}(A) = \text{span}(\mathbf{u}, \mathbf{v})$. Compute $A^T\mathbf{u}$ and $A^T\mathbf{v}$. Explain why, in general, if $\mathbf{v} \in \text{col}(A)$ is non zero, then $A^T\mathbf{v} \neq 0$.
- Describe the left nullspace of A. Why does it only contain the zero vector?
- Construct a 2×3 matrix whose column space is 1-dimensional and whose left nullspace is 1-dimensional.

Remark 8.9. Vectors in the column space of A are perpendicular to vectors in the left nullspace. For example, consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}, \qquad \text{col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

The left nullspace is

$$\text{null}(A^T) = \text{null}\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \right) = \text{null}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}.$$

Taking the dot product of the basis vectors, we find

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2 + 2 = 0,$$

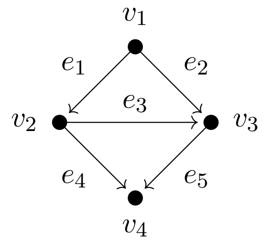
and so every vector in $\text{col}(A)$ is perpendicular to every vector in $\text{null}(A^T)$.

Inquiry 8.10: Consider the vector $\mathbf{v} = \left[\begin{smallmatrix} 1 & 0 & -1 & 0 \end{smallmatrix}\right]^T \in \mathbf{R}^4$, and let $A \in \mathcal{M}_{4\times 4}$.

- Suppose that $\left[\begin{smallmatrix} 1 & 0 & 0 & 0 \end{smallmatrix}\right]^T \in \text{col}(A)$ and $\left[\begin{smallmatrix} 0 & 0 & -1 & 0 \end{smallmatrix}\right] \in \text{null}(A)$. Is $\mathbf{v}$ in either the column space or the nullspace? Does it have to be?
- Find two vectors in $\mathbf{R}^4$ that are perpendicular to $\mathbf{v}$. Explain why this gives you a 4×2 matrix that contains $\mathbf{v}$ in its left nullspace.

Example 8.11. For a practical application of these spaces, consider the following two matrices, both representations of the directed graph below. In A_{inc}, the rows correspond to edges, and the columns correspond to vertices: each row has a -1 for the vertex where the edge starts and a 1 for the vertex where the edge ends. This is called an *incidence matrix*. In A_{adj}, (i, j)-entry is 1 if there is a directed edge from v_i to v_j, and 0 otherwise. This is called the *adjacency matrix*.

$$A_{inc} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad A_{adj} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Bringing the matrix $(A_{inc})^T$ to row reduced echelon form gives informsation about the row space of A_{inc} and the left nullspace of A_{inc}. The linearly independent rows of A_{inc} are the rows corresponding to the edges e_2, e_2, e_4, and these edges form a *spanning tree* of the graph. The dependent row 3, corresponding to edge e_3, is dependent because adding it would create a *cycle* in the graph (among v_1, v_2, v_3), and cycles contain redundant information, so we want to get rid of cycles. Similarly we get a cycle if we add row 5, corresponding to edge e_5, because then we have a cycle of four edges.

8.3 Exercises

Exercise 8.1. Find a basis for the column space, nullspace, row space, and left nullspace of

$$A = \begin{bmatrix} 0 & 1 & a & 0 & a & 0 \\ 0 & 0 & 1 & b & 0 & b \\ 0 & 0 & 0 & 1 & c & c \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

for $a, b, c \in \mathbf{R}$. Do the bases change if any of a, b, c are zero?

Exercise 8.2. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Describe the four fundamental subspaces of A, $A + I$, and $A + A^2$.

Exercise 8.3. Let $\mathbf{u} = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ and $\mathbf{v} = \left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right]$.

1. Construct a 2×4 matrix A for which $\text{col}(A) = \text{span}(\{\mathbf{u}, \mathbf{v}\})$.
2. Find a basis for the column space and row space of $\mathbf{u}\mathbf{v}^T + (\mathbf{u}\mathbf{v}^T)^2$.

Lecture 9: Orthogonality

Chapter 4.1 in Strang's "Linear Algebra"

- Fact 1: Two orthogonal subspaces are orthogonal complements if their dimensions sum up to the dimension of the space they are in.
- Fact 2: The column space is the orthogonal complement to the left nullspace, and the nullspace is the orthogonal complement to the row space.

- Skill 1: Determine if the columns of a matrix are orthogonal.
- Skill 2: Determine if two subspaces are orthogonal.

The vector space pairs column space / nullspace and row space / left nullspace are special because of the relationship of each element of the pair to the other. In this lecture we will generalize this relationship.

9.1 Orthogonal spaces

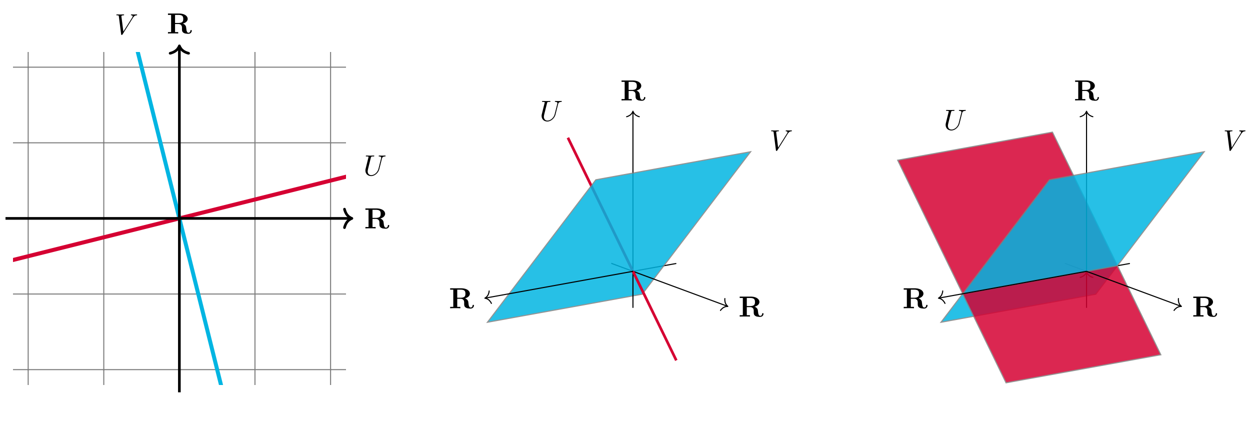
Recall from Lecture 1 that two vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ are *orthogonal* if $\mathbf{u} \bullet \mathbf{v} = \mathbf{u}^T\mathbf{v} = 0$. Note that in this case we have something that looks like the Pythagorean theorem:

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v}) \bullet (\mathbf{u}+\mathbf{v}) = \mathbf{u} \bullet \mathbf{u} + 2\underbrace{\mathbf{u} \bullet \mathbf{v}}_{0} + \mathbf{v} \bullet \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

If $\mathbf{u}, \mathbf{v}$ are orthogonal and both have length 1, then they are called *orthonormal*

Definition 9.1: Two subspaces $U, V \subseteq \mathbf{R}^n$ are *orthogonal* if every pair of vectors $\mathbf{u} \in U, \mathbf{v} \in V$ is orthogonal. We say that "U is orthogonal to V" and "V is orthogonal to U", which both mean the same thing.

Example 9.2. Consider the following subspaces of Euclidean space.



lines at 90° to each other in $\mathbf{R}^2$ are orthogonal

a line coming out of a plane in $\mathbf{R}^3$ at 90° is orthogonal to the plane

two planes in $\mathbf{R}^3$ that "intersect at 90°" are not orthogonal

Two planes "intersecting at 90°" does not make sense, because any angle can be found between the two planes with vectors in them. The planes intersect in a 1-dimensional vector subspace (the x-axis), and the inner product of $[\,1\ 0\ 0\,]^T$ with itself is not zero.

Inquiry 9.3: In Example 9.2 above, the third example with two planes looks like it "should" describe a perpendicular intersection.

- Construct bases for U and for V of two vectors each. Make it so that the bases have a common vector.
- Take the symmetric difference of the two basis sets. What is the angle between the two vectors?

Give a proper description of what the "perpendicular feeling" in the picture is, using bases.

Example 9.4. For $A \in \mathcal{M}_{m\times n}$, the nullspace $\text{null}(A)$ and the row space $\text{row}(A)$ are orthognal to each other. Recall that $\mathbf{x} \in \text{null}(A)$ if $A\mathbf{x} = 0$. Another way of saying this is, for $\mathbf{r}_i \in \mathbf{R}^n$ a row of A, that

$$\begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_m & - \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \bullet \mathbf{x} \\ \mathbf{r}_2 \bullet \mathbf{x} \\ \vdots \\ \mathbf{r}_m \bullet \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and since the row space $\text{row}(A) = \text{span}(\{\mathbf{r}_1, \dots, \mathbf{r}_m\})$, we see that $\mathbf{v} \bullet \mathbf{x} = 0$ for any $\mathbf{v} \in \text{row}(A)$ and for any $\mathbf{x} \in \text{null}(A)$. Applying the same observation to the transpose A^T, we see that the left nullspace of A (which the nullspace of A^T) is orthogonal to the column space of A (which is the row space of A^T).

Inquiry 9.5: This inquiry uses Python, and follows the Pyhton notebook on the course website.

- Generate 100 real-valued vectors $\mathbf{R}^2$, with entries in the range $[0, 5]$. How many pairs are orthogonal? How many have inner product very close to zero?
- Generate 100 integer-valued vectors $\mathbf{R}^2$, with entries in $\{0, 1, 2, 3, 4, 5\}$. How many pairs are orthogonal? How many would you expect to be orthogonal?
- Generalize the previous point to $\mathbf{R}^3$.

Remark 9.6. To check that two vector spaces are orthogonal, it suffices to check that every pair of elements $\mathbf{u} \in B$, $\mathbf{v} \in B'$ are orthogonal, for B a basis of U and B' a basis for V.

We now consider orthogonality in the context of particular matrices.

Example 9.7. The matrix $R_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ is called the *rotation matrix*, since the angle between $\mathbf{v} \in \mathbf{R}^2$ and $R_\theta \mathbf{v} \in \mathbf{R}^2$ is exactly θ. The columns of R_θ are orthogonal, as

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = -\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta) = 0,$$

for any angle θ. The columns are also orthonormal, as

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \cos^2(\theta) + \sin^2(\theta) = 1,$$

$$\begin{bmatrix} -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = \sin^2(\theta) + \cos^2(\theta) = 1.$$

Example 9.8. Consider a matrix $A \in \mathcal{M}_{3\times 6}$ as below. It does not have all orthogonal rows and columns, as row reduction shows we have only two pivots, meaning the row rank = column rank is 2:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 6 \\ 2 & 4 & 8 & 10 & 12 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^T = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 4 & 8 \\ 4 & 5 & 10 \\ 5 & 6 & 12 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

That is, columns 1 and 3 of A describe the 2-dimensional column space orthogonal to the 1-dimensional left nullspace of row 3. Analogously, columns 2,4,5 of A describe the 3-dimensional nullspace orthogonal to the row space of rows 1 and 2 of A:

$$\operatorname{col}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} \right\} \quad \text{is orthogonal to} \quad \operatorname{null}(A^T) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\},$$

$$\operatorname{null}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{is orthogonal to} \quad \operatorname{row}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 6 \end{bmatrix} \right\}.$$

We are left with a 2×2 *invertible submatrix* of A, hiding in the intersection of the pivot rows and pivot columns. This submatrix is important for finding left and right inverses of non-square matrices, and for *singular value decomposition*, which we will see later in the course.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 6 \\ 2 & 4 & 8 & 10 & 12 \end{bmatrix}, \qquad A_{inv} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}.$$

9.2 Orthogonal relationships

Definition 9.9: If two subspaces $U, V \subseteq \mathbf{R}^n$ are orthogonal and $\dim(U) + \dim(V) = n$, then each is the *orthogonal complement* of the other in $\mathbf{R}^n$. That is, U is the orthogonal complement of V, written $U = V^\perp$, and V is the orthogonal complement of U, written $V = U^\perp$.

Remark 9.10. Recall the concept of *codimension* from Definition 7.18. The codimension of a space is equal to the dimension of its orthogonal complement. That is, $\operatorname{codim}(U) = \dim(U^\perp)$.

Remark 9.11. It follows that, whenever we have orthogonal complements $U = V^\perp$, with $U, V \subseteq \mathbf{R}^n$ subspaces, then:

- $U + V = \mathbf{R}^n$, or in other words,
- any $\mathbf{x} \in \mathbf{R}^n$ can be expressed as a sum $\mathbf{x} = \mathbf{u} + \mathbf{v}$ of two elements, $\mathbf{u} \in U$ and $\mathbf{v} \in V$.

Theorem 9.12. *Let U, V be subspaces of $\mathbf{R}^n$. Then*

1. $(U^\perp)^\perp = U$
2. $(U + V)^\perp = U^\perp \cap V^\perp$
3. $(U \cap V)^\perp = U^\perp + V^\perp$

Proof. We only prove the second point, you will prove the other points in your homework. Recall that $U + V = \{\mathbf{u} + \mathbf{v} \ : \ \mathbf{u} \in U, \mathbf{v} \in V\}$. Take $\mathbf{u} \in U$, $\mathbf{v} \in V$, and $\mathbf{x} \in (U + V)^\perp$. To see that $(U + V)^\perp \subseteq U^\perp \cap V^\perp$, notice that $\mathbf{u}, \mathbf{v} \in U + V$, hence

$$\mathbf{u} \bullet \mathbf{x} = 0 \implies \mathbf{x} \in U^\perp, \qquad \mathbf{v} \bullet \mathbf{x} = 0 \implies \mathbf{x} \in V^\perp,$$

and so $\mathbf{x} \in U^\perp \cap V^\perp$. Since the vectors were arbitrary, we get that $(U+V)^\perp \subseteq U^\perp + V^\perp$. To see that $U^\perp \cap V^\perp \subseteq (U+V)^\perp$, take $\mathbf{y} \in U^\perp \cap V^\perp$, which means that both $\mathbf{y} \in U^\perp$ and $\mathbf{y} \in V^\perp$. Consider the arbitrary element $\mathbf{u}+\mathbf{v} \in U+V$, for which

$$\mathbf{y} \bullet (\mathbf{u}+\mathbf{v}) = \mathbf{y} \bullet \mathbf{u} + \mathbf{y} \bullet \mathbf{v} = 0 + 0 = 0,$$

meaning that $\mathbf{y} \in (U+V)^\perp$. Again, since the vectors are arbitrary, it follows that $U^\perp \cap V^\perp \subseteq (U+V)^\perp$. Combining these two statements, we get that $(U+V)^\perp = U^\perp \cap V^\perp$. □

Example 9.13. Combining Example 9.4 and the rank-nullity theorem from Lecture 8, for $A \in \mathcal{M}_{m\times n}$ we see that

- the nullspace and row space are orthogonal complements: $\text{null}(A) = \text{row}(A)^\perp$
- the left nullspace and column space are orthogonal complements: $\text{null}(A^T) = \text{col}(A)^\perp$

That is, along with Remark 9.11, any $\mathbf{x} \in \mathbf{R}^n$ can be written as a sum $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$, where $\mathbf{x}_r \in \text{row}(A)$ and $\mathbf{x}_n \in \text{null}(A)$. It follows that no row of A can be in the nullspace of A.

Inquiry 9.14: Let V be a vector space and $U, W \subseteq V$ subpaces, with $U = W^\perp$.

- If $\dim(V) = n$ and $\dim(U) = \dim(W)$, explain what k must be, in terms of n.
- You are given that $U = \text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_i)$ and $W = \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_j)$. Explain the relationship between i, j, n.

We finish this lecture with an observation about bases of $\mathbf{R}^n$.

Remark 9.15. Recall that to be a basis of $\mathbf{R}^n$, a set of vectors has to be linearly independent and had to span $\mathbf{R}^n$. It follows that:

- If a set of n vectors is linearly independent, it spans $\mathbf{R}^n$.
- If n vectors span $\mathbf{R}^n$, they must be linearly independent.

The second fact comes from considering an $n \times n$ matrix A whose columns span $\mathbf{R}^n$, or equivalently, where for every $\mathbf{b} \in \mathbf{R}^n$ there is a unique solution $\mathbf{x}$ in $A\mathbf{x} = \mathbf{b}$. If we argue that the columns are linearly dependent, then there must be at least one special solution, and so infinitely many solutions to $A\mathbf{x} = \mathbf{b}$, but this contradicts what we originally assumed.

9.3 Exercises

Exercise 9.1. Confirm the observation from Remark 9.6. That is, let $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a basis for a vector space $U \subseteq \mathbf{R}^n$, and let $B' = \{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ be a basis for a vector space $V \subseteq \mathbf{R}^n$. If you know that $\mathbf{u}_i \cdot \mathbf{v}_j = 0$ for all i, j, check that $\mathbf{u} \bullet \mathbf{v} = 0$ for arbitrary elements $\mathbf{u} \in U$ and $\mathbf{v} \in V$.

Exercise 9.2. Check that the claim about the angle between $\mathbf{v}$ and $R_\theta \mathbf{v}$ from Example 9.7 is indeed true.

Exercise 9.3. Let $A \in \mathcal{M}_{m\times n}$. Show that there is a bijective function $f \colon \text{row}(A) \to \text{col}(A)$. Hint: use orthogonality and the decomposition of vectors described in Example 9.13.

Exercise 9.4. Let U, V be subspaces of $\mathbf{R}^n$.

1. Show that $(U^\perp)^\perp = U$.
2. Show that $(U \cap V)^\perp = U^\perp + V^\perp$.

3. Suppose there exist matrices A, B with $U = \mathrm{col}(A)$ and $V = \mathrm{col}(B)$. Find a matrix C for which $\mathrm{null}(C) = (U + V)^{\perp}$.
   Hint: construct C as a block matrix.

Exercise 9.5. Consider the following two planes, as subspaces of $\mathbf{R}^3$:

$$P_1 = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ 3x_1 - 4x_2 + x_3 = 0\},$$
$$P_2 = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ 5x_1 - 10x_3 = 0\}.$$

1. Find normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ to the planes P_1 and P_2, respectively.

2. Find bases B_1 and B_2 for the planes P_1 and P_2, respectively.
   Hint: the basis of a plane is the nullspace of the defining equation.

3. Construct a 2×3 matrix A_1 whose row space is P_1. Show that the nullspace of A_1 is the span of $\mathbf{n}_1$.

4. Construct a 3×2 matrix A_2 whose column space is P_2. Show that the left nullspace of A_2 the span of $\mathbf{n}_2$.

Lecture 10: Projections

Chapter 4.2 in Strang's "Linear Algebra"

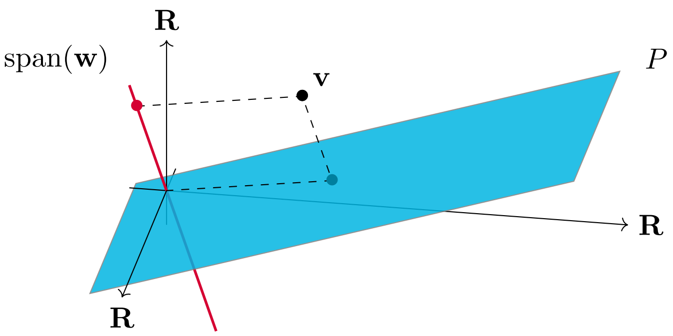
- Fact 1: A projection of a vector is another vector.
- Fact 2: Projections are "best approcximations"

- Skill 1: Compute the projection of vectors onto subspaces.

We continue our study of orthogonality by describing how it affects arbitrary vectors, not just ones in the vector subspaces being considered.

10.1 Projecting onto lines

To *project* a vector $\mathbf{v} \in \mathbf{R}^3$ onto some other vector $\mathbf{w} \in \mathbf{R}^3$ (or onto some plane P going through the origin), means to create a new vector that points in the same direction as $\mathbf{w}$ (or lies in P), and is "as close as possible" to the first vector $\mathbf{v}$.



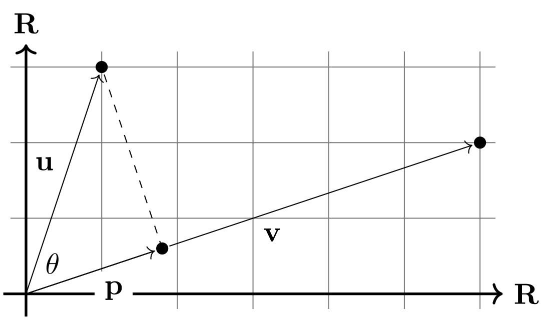
Since both $\mathbf{w}$ and P are subspaces of $\mathbf{R}^3$, projections can be understood in (at least) two ways:

1. the projection of $\mathbf{v}$ is the part of $\mathbf{v}$ that lies in the subspace to which you are projecting
2. the projection of $\mathbf{v}$ produces another vector $\mathbf{v}'$, so projecting is simply multiplying by some appropriate matrix A: $A\mathbf{v} = \mathbf{v}'$

Both of these approaches are correct.

Example 10.1. Projecting $\mathbf{v} \in \mathbf{R}^3$ onto the y-axis is multiplying $\mathbf{v}$ by $\begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}$. Projecting $\mathbf{v}$ onto the xy-plane is multiplying $\mathbf{v}$ by $\begin{bmatrix} 1&0&0\\0&1&0\\0&0&0 \end{bmatrix}$.

In general, projecting a vector $\mathbf{u}$ onto a vector $\mathbf{v}$ uses the formula for the angle between them, from Proposition 1.17. Given two such arbitrary vectors, we want to compute the vector $\mathbf{p}$, which goes in the direction of $\mathbf{v}$, and is one side of a right triangle with $\mathbf{u}$ as hypotenuse.



$$\cos(\theta) = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Since the hypotenuse has length $\|\mathbf{u}\|$, the adjacent, which is $\mathbf{p}$, must have length $\frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|}$. The vector $\mathbf{v}$ may not have unit length, but the vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ does, and it goes in the same direction as $\mathbf{v}$. Hence $\mathbf{p}$ may be expressed as

$$\frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|}\cdot\frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}=\frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v}.$$

Definition 10.2: The *projection* of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\text{proj}_{\mathbf{v}}(\mathbf{u})=\frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v}. \tag{3}$$

The difference $\mathbf{u}-\text{proj}_{\mathbf{v}}(\mathbf{u})$ is called the *error vector*.

Example 10.3. We note two trivial examples of projections.

- Projecting $\mathbf{u}$ onto a line which is orthogonal to $\mathbf{u}$ gives the zero vector. This makes sense, because $\mathbf{u}\bullet\mathbf{v}=0$ for all $\mathbf{v}$ in this line. In this case the error vector is equal to $\mathbf{u}$.
- Projecting $\mathbf{u}$ onto the line on which $\mathbf{u}$ already lies gives back $\mathbf{u}$. This also makes sense, because the line is all vectors $c\mathbf{u}$, for $c\in\mathbf{R}$, and for $\mathbf{v}=c\mathbf{u}$, the expression $\frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}$ becomes $\frac{1}{c}$, and $\frac{1}{c}\mathbf{v}=\mathbf{u}$. In this case the error vector is the zero vector.

Considering the dot product as multiplication of matrices, Equation (3) becomes

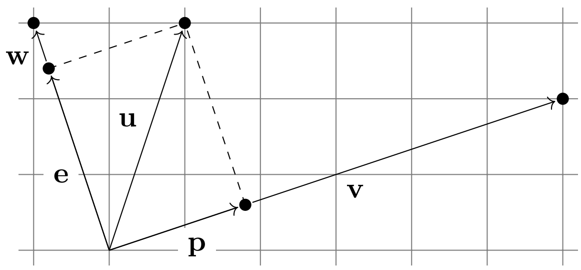
$$\frac{\mathbf{u}^T\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v}=\frac{\mathbf{v}^T\mathbf{u}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v}=\mathbf{v}\frac{\mathbf{v}^T\mathbf{u}}{\mathbf{v}\bullet\mathbf{v}}=\frac{\mathbf{v}\mathbf{v}^T\mathbf{u}}{\mathbf{v}\bullet\mathbf{v}}=\underbrace{\frac{1}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v}\mathbf{v}^T}_{P}\mathbf{u}. \tag{4}$$

The matrix P is the rank one *projection matrix*. The idea for it being rank one is that the projection goes to a 1-dimensional subspace, a line.

Inquiry 10.4: This inquiry continues the ideas from Example 10.3 above

- Explain what properties of scalar, vector, or matrix operations are being used for each equality in Equation (4).
- Let $P=\left[\begin{smallmatrix}a&b\\b&c\end{smallmatrix}\right]\in\mathcal{M}_{2\times 2}$ be a symmetric matrix. Using Equation (4), explain what conditions must be true for $a,b,c\in\mathbf{R}$ for P to represent a projection matrix.

Remark 10.5. The error vector $\mathbf{e}=\mathbf{u}-\mathbf{p}$ from Definition 10.2 is also a type of projection, but onto a different vector, one that is orthogonal to $\mathbf{v}$ and $\mathbf{p}$.



To get a matrix for the projection of $\mathbf{u}$ onto $\mathbf{w}$, we want the result to be $\mathbf{e}=\mathbf{u}-\mathbf{p}$. Since $\mathbf{p}=P\mathbf{u}$, we quickly see that $\mathbf{e}=(I-P)\mathbf{u}$. Hence the projection matrix is $I-P$.

10.2 Projecting onto subspaces

Next we consider the more general situation of projection a vector onto a subspace. Since all vector spaces have a spanning set, we consider a subspace to be a span of vectors. Combining these vectors as columns of a matrix, we get the column space.

Definition 10.6: Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq \mathbf{R}^n$, and let A be the matrix with these vectors as its columns. For any $\mathbf{u} \in \mathbf{R}^n$, the *projection* of $\mathbf{u}$ onto V is the vector

$$\text{proj}_V(\mathbf{u}) = \underbrace{A(A^TA)^{-1}A^T}_{P}\mathbf{u}.$$

We assume the $\mathbf{v}_i$ are linearly independent, as otherwise A^TA does not have an inverse. If the v_i are not independent, remove the vectors that depend on others (this does not change the span).

The motivation for this expression is slightly more tedious, and comes from observing that for $\mathbf{p} = A\mathbf{x}$ the projection (for some appropriate $\mathbf{x}$), the vector $\mathbf{u} - A\mathbf{x}$ is orthogonal to the column space of A.

Remark 10.7. Since $V^\perp$ is the orthogonal complement of V, by Remark 9.11, every $\mathbf{u} \in \mathbf{R}^n$ can be expressed as $\mathbf{u} = \mathbf{v} + \mathbf{v}'$, where $\mathbf{v} \in V$ and $\mathbf{v}' \in V^\perp$. Since matrix multiplication is linear, and using the trivial projections from Example 10.3, it follows that

$$\begin{aligned}\text{proj}_V(\mathbf{u}) &= \text{proj}_V(\mathbf{v} + \mathbf{v}') = \text{proj}_V(\mathbf{v}) + \text{proj}_V(\mathbf{v}') = \mathbf{v} + 0 = \mathbf{v},\\ \text{proj}_{V^\perp}(\mathbf{u}) &= \text{proj}_{V^\perp}(\mathbf{v} + \mathbf{v}') = \text{proj}_{V^\perp}(\mathbf{v}) + \text{proj}_{V^\perp}(\mathbf{v}') = 0 + \mathbf{v}' = \mathbf{v}',\end{aligned}$$

and so we always have $\mathbf{v} = \text{proj}_V(\mathbf{v}) + \text{proj}_{V^\perp}(\mathbf{v})$ for any $\mathbf{v} \in \mathbf{R}^n$. This gives a matrix for projecting onto the orthogonal complement, as

$$\text{proj}_{V^\perp}(\mathbf{u}) = \mathbf{u} - \text{proj}_V(\mathbf{u}) = \mathbf{u} - A(A^TA)^{-1}A^T\mathbf{u} = \underbrace{(I - A(A^TA)^{-1}A^T)}_{P}\mathbf{u}.$$

Inquiry 10.8: Let $V = \mathbf{R}^2$, choose two perpendicular vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{R}^2$, and compute the two projection matrices $P_1 = \text{proj}_{\text{span}(\mathbf{v}_1)}$ and $P_2 = \text{proj}_{\text{span}(\mathbf{v}_2)}$. Let $\mathbf{w} \in \mathbf{R}^2$ be a non-trivial linear combination of both of the vectors.

- Explain why $P_1P_2\mathbf{w} = P_2P_1\mathbf{w} = 0$. Is $P_1P_2 = P_2P_1 = 0$?
- Explain why $P_1P_1\mathbf{w} = P_1\mathbf{w}$ and $P_2P_2\mathbf{w} = P_2\mathbf{w}$. Is $P_1P_1 = P_2P_2 = I$?

See Exercise 10.1 for more guidance.

10.3 Exercises

Exercise 10.1. Show that projecting twice onto a line is the same as projecting once. That is, if P is the projection matrix from Equation (4), show that $P^2 = P$.

Exercise 10.2. Let $\mathbf{v} = (1, 1, 1) \in \mathbf{R}^3$.

⋈ 1. Take random vectors in the unit square in $\mathbf{R}^3$, and plot the average error, up until 1000 vectors, when projecting to $\mathbf{v}$.
2. What does this number converge to?
3. **Bonus:** Prove this limit.

Exercise 10.3. Find the projection of $\mathbf{v} = (-3, -1, 6)$ onto the plane $3x + 4y - 9z = 0$ and its normal vector.

Exercise 10.4. Let $\mathbf{v} = (x, y, z, w)$.

1. What matrix M projects $\mathbf{v}$ onto the xy-plane to produce $(x, y, 0, 0)$? That is, find M for $M\mathbf{v} = (x, y, 0, 0)$.

2. What matrix N cycles the axes to produce (w, x, y, z)? That is, find N for $N\mathbf{v} = (w, x, y, z)$.

Exercise 10.5. The set $U \subseteq \mathbf{R}^n$ is a subspace with basis $\mathbf{u}_1, \ldots, \mathbf{u}_k$. These basis vectors are the columns of the $n \times k$ matrix A. For any $\mathbf{v} \in \mathbf{R}^n$, define the *reflection* of $\mathbf{v}$ in U to be the vector

$$\text{refl}_U(\mathbf{v}) := \mathbf{v} - 2\text{proj}_{U^\perp}(\mathbf{v}).$$

1. Construct the matrix of refl_U.

2. Show that refl_U preserves length, that is, show that $\|\text{refl}_U(\mathbf{v})\| = \|\mathbf{v}\|$ for all $\mathbf{v} \in \mathbf{R}^n$.

Lecture 11: The least squares approximation

Chapter 4.3 in Strang's "Linear Algebra"

- Fact 1: The least squares approximation is a vector $\hat{\mathbf{x}}$ that is "the closest solution to" $A\mathbf{x} = \mathbf{b}$ when $\mathbf{b} \notin \text{col}(A)$
- Fact 2: If $A\mathbf{x} = \mathbf{b}$ does not have a solution, then $A^T A\mathbf{x} = \mathbf{b}$ will have a solution, as long as the rows of A are linearly independent.

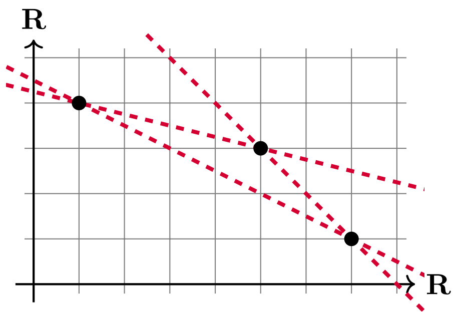
- Skill 1: Find the least squares solution to a matrix equation with no solution.
- Skill 2: Find the degree-d polynomial that approximates a collection of points in $\mathbf{R}^2$.

One of the main applications of projections is finding the the closest solution to a linear system that has no exact solution.

11.1 Least squares for lines

When given points in the plane $\mathbf{R}^2$, it is often assumed there is some underlying relationship among the points. To discover this relationship from the points, some approximation must be made, becaused the points are never arramnged in a neat pattern.

Example 11.1. Consider the points $(1, 4)$, $(7, 1)$, $(5, 3) \in \mathbf{R}^2$. Is there a line $y = ax + b$ goes through all of them? If yes, which one is it? If no, why?



There is no such line, because any two of the points determine a line that does not intersect the third point. We are equivalently asking for a solution to three equations, or to a linear system.

$$\begin{aligned} 4 &= a + b \\ 1 &= 7a + b \\ 3 &= 5a + b \end{aligned} \qquad \underbrace{\begin{bmatrix} 1 & 1 \\ 7 & 1 \\ 5 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 7 & 1 & 1 \\ 5 & 1 & 3 \end{array}\right] \xrightarrow{G.E.} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -6 & -27 \\ 0 & 0 & 1 \end{array}\right]$$

Note that $[\,4\;1\;3\,]^T$ is not in the column space of the matrix A, since the auigmented matrix by Gaussian elimination gives the contradictory equation $0 = 1$ in the last row. However, we still want to find a line that is "as close as possible", and projections help us do that.

Remark 11.2. Above we had a matrix equation $A\mathbf{x} = \mathbf{b}$ for which $\mathbf{b} \notin \text{col}(A)$. However, we can project $\mathbf{b}$ onto $\text{col}(A)$, which will guarantee a solution. That is, we can always write $\mathbf{b} = \mathbf{p} + \mathbf{e}$, where $\mathbf{p} \in \text{col}(A)$ and $\mathbf{e}$ is orthogonal to $\text{col}(A)$.

Definition 11.3: Let $A \in \mathcal{M}_{m\times n}$ and $\mathbf{b} \in \mathbf{R}^m$, with $\mathbf{b} = \mathbf{p} + \mathbf{e}$ and $\mathbf{p} \in \text{col}(A)$. The *least squares* solution to $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ that, equivalently,

- makes the distance between $A\mathbf{x}$ and $\mathbf{b}$ as small as possible
- makes the number $\|A\mathbf{x} - \mathbf{b}\|$ as small as possible
- is the solution to $A\mathbf{x} = \mathbf{p}$

In practice, we minimize $\|A\mathbf{x} - \mathbf{b}\|^2$ instead of $\|A\mathbf{x} - \mathbf{b}\|$, since square roots are hard to deal with. It does not matter which expresion we minimize, because $a < b$ iff $a^2 < b^2$ for a, b nonnegative. The first approach to finding the least squares solution is to use *calculus*, because that is how to find the minimum of a quadratic function.

Example 11.4. Using the equation $A\mathbf{x} = \mathbf{b}$ fom Example 11.1, we have

$$\|A\mathbf{x} - \mathbf{b}\|^2 = \left\| \begin{bmatrix} 1 & 1 \\ 7 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} a+b \\ 7a+b \\ 5a+b \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} a+b-4 \\ 7a+b-1 \\ 5a+b-3 \end{bmatrix} \right\|^2,$$

which simplifies to

$$M(a,b) = (a+b-4)^2 + (7a+b-1)^2 + (5a+b-3)^2. \tag{5}$$

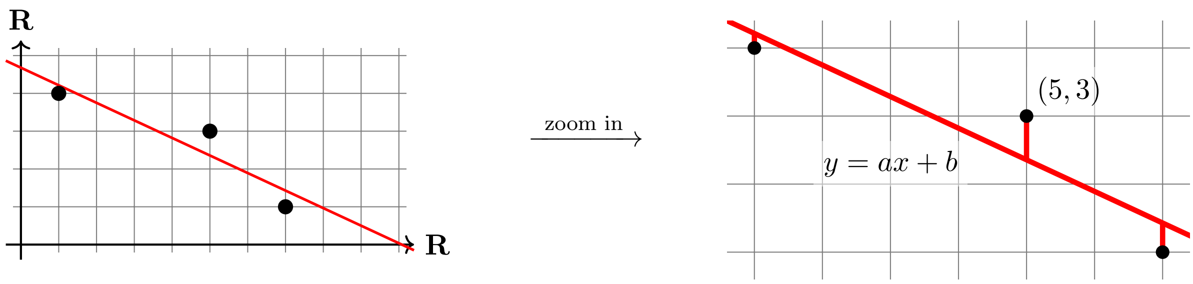
To find its minimum, we take the derivative. Since this is a function in two variables, we have two derivatives to take.

$$\begin{aligned} \frac{\partial M}{\partial a} &= 2(a+b-4) + 2(7a+b-1)(7) + 2(5a+b-3)(5) = 150a + 26b - 52 \\ \frac{\partial M}{\partial b} &= 2(a+b-4) + 2(7a+b-1) + 2(5a+b-3) = 26a + 6b - 16 \end{aligned}$$

Having these derivatives be zero produces a new matrix equation to solve:

$$\begin{bmatrix} 150 & 26 \\ 26 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 52 \\ 16 \end{bmatrix} : \qquad \begin{bmatrix} 150 & 26 & 52 \\ 26 & 6 & 16 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -\frac{13}{28} \\ 0 & 1 & \frac{131}{28} \end{bmatrix}$$

We now see the line $y = -\frac{13}{28}x + \frac{131}{28}$ is the best approximation:



The vertical distances from the points to the line have been minimized. Indeed, for example with $(5, 3)$, minimizing the vertical distance between it and the line $y = ax + b$ means making the value

$$\|(5, 5a+b) - (5,3)\|^2 = \|(5-5, 5a+b-3)\|^2 = (5-5)^2 + (5a+b-3)^2 = (5a+b-3)^2$$

as small as possible, which is exactly the third term in $M(a, b)$ from Equation (5).

Inquiry 11.5: This question is about least squares approximations.

- You already know that two points in $\mathbf{R}^2$ define a unique line. Explain why the method provided in Example 11.4 will produce this unique line.
- Suppose you are given three points above are above each other, such as $(1, 2)$, $(1, 4)$, and $(1, 6)$. Explain why the described method will fail. How would you fix it?
- Explain why there is no problem with the task above if a fourth point is added at a different x-value.

Remark 11.6. The "distance" from a point to the line can be throught of as the shortest length - not always the vertical distance. This is sometimes called the *perpendicular* distance, and will be solved by the method presented later in Lecture 22.

The second approach is to observe that for $\mathbf{b} = \mathbf{p} + \mathbf{e}$, the error vector $\mathbf{e}$ is in the left nullspace of A, since the column space and left nullspace are orthogonal complements.

Theorem 11.7. *Let $A \in \mathcal{M}_{m\times n}$. If $\boldsymbol{b} \notin \text{col}(A)$, then*

1. *the equation $A\boldsymbol{x} = \boldsymbol{b}$ has no solution, and*
2. *the equation $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$ does have a solution.*

The justification for the second point of this statement is given in the folowing inquiry.

Inquiry 11.8: This inquiry explains the reasoning behind Theorem 11.7. Let $A\mathbf{x} = \mathbf{b}$ be a matrix equation with $\mathbf{b} \notin \text{col}(A)$, and $\mathbf{b} = \mathbf{p} + \mathbf{e}$, where $\mathbf{p} = \text{proj}_{\text{col}(A)}(\mathbf{b})$.

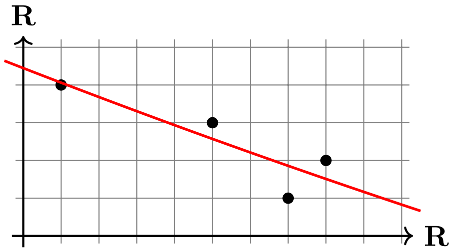
- Explain why the error vector $\mathbf{e} \in \text{null}(A^T)$. *Hint: use orthogonal complements.*
- What does $A^T\mathbf{b}$ simplify to, when $\mathbf{b}$ is replaced by $\mathbf{p} + \mathbf{e}$? Use what you showed above.
- Convince yourself that $A^T\mathbf{p} \in \text{col}(A^T)$. Explain why this means that $A^T\mathbf{p} \notin \text{null}(A)$.
- Show that if $\mathbf{x} \in \text{null}(A)$, then $\mathbf{x} \in \text{null}(A^TA)$. *Hint: use orthogonal complements.*
- Show that if $\mathbf{x} \in \text{null}(A^TA)$, then $\mathbf{x} \in \text{null}(A)$. *Hint: use the positive definiteness of the norm.*
- Put everything together to get that $A^T\mathbf{p} \in \text{col}(A^TA)$. Explain why this means that $A^TA\mathbf{x} = A^T\mathbf{b}$ has a solution.

For points 4 and 5, see Exercise 11.2 for more guidance.

11.2 Least squares for higher degree polynomials

Suppose we want to generalize the previous section, and find a quadratic function that goes through three points in the plane $\mathbf{R}^2$. Quadratics have the form $y = ax^2 + bx + c$, so there are three variables a, b, c that need to be found.

Example 11.9. Three points always have a unique quadratic going through them (which can be found by back-substitution), so we add another point $(8,2)$ for increased difficulty.



$$\begin{aligned} 4 &= a+b+c \\ 1 &= 49a+7b+c \\ 3 &= 25a+5b+c \\ 2 &= 64a+8b+c \end{aligned} \qquad \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 49 & 7 & 1 \\ 25 & 5 & 1 \\ 64 & 8 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix}}_{\mathbf{b}}$$

The process then is very similar, except we have three variables:

$$M(a,b,c) = \|A\mathbf{x}-\mathbf{b}\|^2 = (a+b+c-4)^2+(49a+7b+c-1)^2+(25a+5b+c-3)^2+(64a+8b+c-2)^2.$$

Taking the derivative in all three variables gives

$$\begin{aligned} \frac{\partial M}{\partial a} &= 14246a+1962b+278c-512, \\ \frac{\partial M}{\partial b} &= 1962a+278b+42c-84, \\ \frac{\partial M}{\partial c} &= 278a+42b+8c-20, \end{aligned}$$

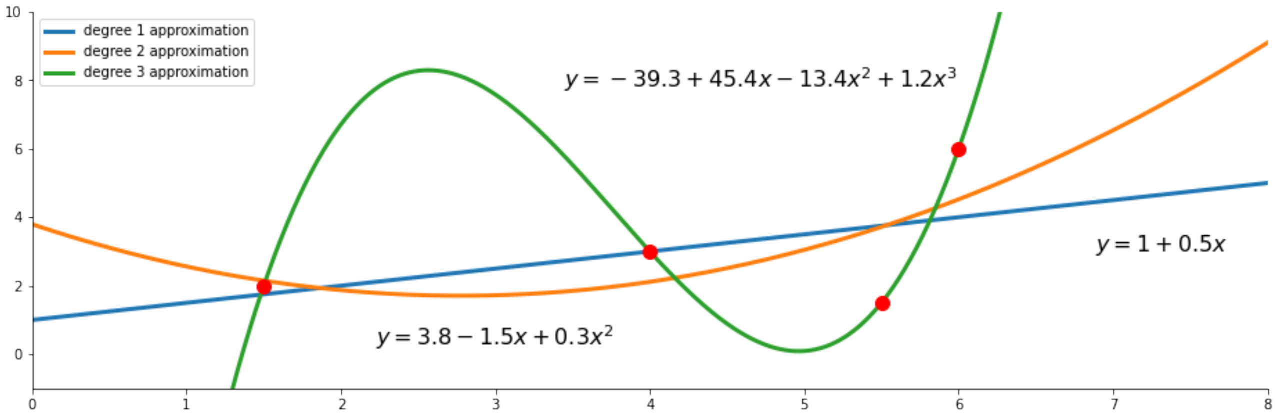
which, when placed into a system, leads to the solutions $a=\frac{1}{372}$, $b=-\frac{241}{620}$, $c=\frac{2068}{465}$, as shown in the plot above.

Definition 11.10: Let $\mathbf{p}_1=(x_1,y_1),\dots,\mathbf{p}_n=(x_n,y_n)\in\mathbf{R}^2$. The degree-$d$ polynomial $a_0+a_1x+a_2x^2+\cdots+a_dx^d$ that approximates the points $\mathbf{p}_i$ is the least squares solution to the matrix equation

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^d \\ 1 & x_2 & x_2^2 & \cdots & x_2^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^d \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

The matrix on the left is called the *Vandermonde* matrix. This is the same as we used before, but with rows rearranged (the solution will be the same).

Example 11.11. Suppose that we have four points in the plane. The degree 1, 2, and 3 appoximations to the four points are given below. Note that individually, the points do not get close to the higher degree approximations, but the degree 3 approximation does go through all of them.



Inquiry 11.12: Let $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbf{R}^2$.

- Explain what condition must hold for the degree-$(n-1)$ polynomial given by Definition 11.10 to go through every one of the points perfectly.
- Let $\varepsilon > 0$ be a very small value, such as $\frac{1}{1000}$. Let $n = 8$, with $\mathbf{p}_i = (i, \varepsilon)$ for $i = 1, 2, 3, 4$, and $\mathbf{p}_i = (i, -\varepsilon)$ for $i = 5, 6, 7, 8$. Explain which degree-d approximation, for $d = 1, \ldots, 7$, is the "best" approximation for these points.

11.3 Exercises

Exercise 11.1. Using the setup from Example 11.1, finished in Example 11.4, to come to the same conclusion (that is, the same best fit linear equation), but use the projection matrix instead of partial derivatives.

Exercise 11.2. Let $A \in \mathcal{M}_{m\times n}$. Show that A and $A^T A$ have the same nullspace.

Exercise 11.3. Consider the set of six points $P = \{p_1, \ldots, p_6\} \subseteq \mathbf{R}^2$, with:

$$p_1 = (-1, 3),\ p_2 = (4, 6),\ p_3 = (3, 1),\ p_4 = (-2, -3),\ p_5 = (6, -7),\ p_6 = (-6, 4).$$

1. Either using the projection matrix or partial derivatives, find the line $y = ax + b$ that is the least squares approximation to the points.
2. Find a point $p_7 \in \mathbf{R}^2$ such that the least squares approximation to P is the same as to $P \cup \{p_7\}$. *Hint: Don't redo all your work! Use an observation from partial derivatives.*
3. Let $c \in \mathbf{R}$. Find a point $p_8 \in \mathbf{R}^2$ such that the least squares approximation to $P \cup \{p_8\}$ has slope c.

Exercise 11.4. ⋈ Write a function in Python that takes two inputs:

- a list of points in $\mathbf{R}^2$,
- a positive integer d,

and returns the degree-d least squares approximation to the input points. You may use the `solve` command from `numpy.linalg` or `scipy.linalg` .

Exercise 11.5. Consider the following collection of four points $P = \{p_1, p_2, p_3, p_4\} \subseteq \mathbf{R}^3$:

$$p_1 = (1, -2, -4), \quad p_2 = (0, 5, 5), \quad p_3 = (-6, -7, 2), \quad p_4 = (1, 4, -1).$$

1. Generalize the least squares approach and find the closest plane H in $\mathbf{R}^3$ to the points in P (instead of the closest line in $\mathbf{R}^2$).
2. Project the points in P onto the plane H from part 1. *Warning: The plane H will not go through the origin. You need to shift everything first.*

Exercise 11.6. Find the least squares degree 1,2,3,4 polynomials that approximate the points

$$(-7, 2),\ (-6, -2),\ (-2, -1),\ (0, 3),\ (3, 0),\ (4, 1).$$

Plot all the functions and points together to confirm that the higher degree polynomials are better approximations to the points.

Exercise 11.7. Any line in $\mathbf{R}^3$ may be given (not uniquely) by $\ell(t) = (a_1, a_2, a_3)t + (b_1, b_2, b_3)$.

1. Given two such arbitrary lines, find the location of the points on each which minimize the distance between them.
2. ⋈ Take 1000 pairs of such random lines and find the average and standard deviation of the minimum distance between the lines.

Lecture 12: The Gram–Schmidt process

Chapter 4.4 in Strang's "Linear Algebra"

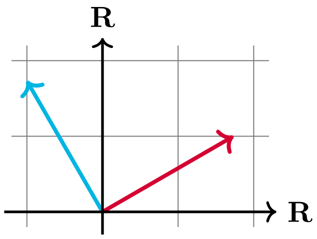
- Fact 1: Every basis can be made into an orthonormal basis.
- Fact 2: The result of the Gram–Schmidt process depends on the order of the vectors input.

- Skill 1: Apply the Gram–Schmidt process to a set of vectors.
- Skill 2: Factorize a matrix A with linearly independent columns as $A = QR$, with A having orthonormal columns.

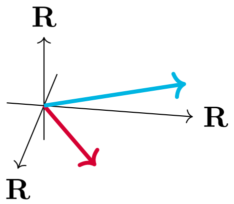
12.1 Orthonormalizing a basis

We previously saw orthogonality and orthonormality in Section 9. We revisit it here from the perspective of bases. Recall that for a set of vectors $B = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ to be *orthonormal*, they need to be orthogonal (that is, $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$), and they need to be of unit length (that is $\|\mathbf{v}_i\| = 1$ for all i).

Remark 12.1. Placing orthonormal vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbf{R}^n$ as columns in a matrix A will always give $A^T A = I$. For example, taking two orthonormal vectors $\begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$ and $\begin{bmatrix} -1/2 \\ \sqrt{3}2 \end{bmatrix}$ in $\mathbf{R}^2$ as columns of a matrix will show this property, as well as when we consider them as lying in the xy-plane of $\mathbf{R}^3$.



$$\underbrace{\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\underbrace{\begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \\ 0 & 0 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The key idea here is that even though there are many different pairs of orthonormal vectors, they all have the common property that they multipy with their transpose to the identity matrix.

Example 12.2. We have already seen the *rotation* matrix $R_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ from Example 9.7 in Lecture 9 has orthonormal columns:

$$R_\theta^T R_\theta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Every single *permutation* matrix also has orthogonal columns:

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{P^T} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Remark 12.3. Whenever $A \in \mathcal{M}_{m\times n}$ has orthonormal columns, the lengths of $\mathbf{v}$ and $A\mathbf{v}$ are the same, for any $\mathbf{v} \in \mathbf{R}^n$. This follows directly from Remark 12.1:

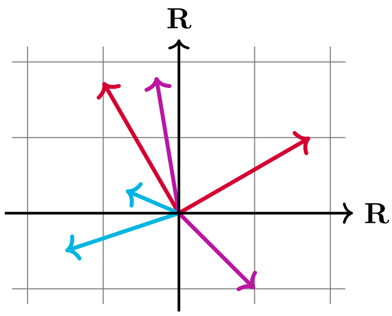
$$\|A\mathbf{v}\|^2 = (A\mathbf{v}) \bullet (A\mathbf{v}) = (A\mathbf{v})^T(A\mathbf{v}) = (\mathbf{v}^T A^T)A\mathbf{v} = \mathbf{v}^T(A^T A)\mathbf{v} = \mathbf{v}^T I\mathbf{v} = \mathbf{v}^T\mathbf{v} = \mathbf{v} \bullet \mathbf{v} = \|\mathbf{v}\|^2.$$

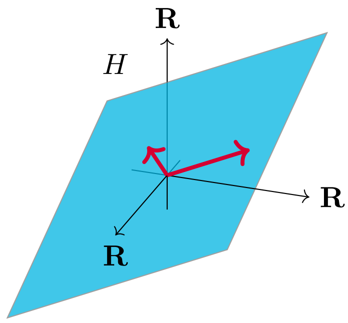
Inquiry 12.4: Let V be a vector space, and $U \subseteq V$ a subspace with basis vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$. Suppose that the basis vectors are orthonormal.

- How does the formula for the projection matrix P simplify, when projecting onto U? See Definition 10.6.
- Let $\mathbf{v} \in V$. Express the projection $\text{proj}_U(\mathbf{v})$ as a linear combination of the basis vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ of U.

We are considering all the impacts of having an orthonormal basis, because a very helpful simplification to many problems is to have an orthnormal basis. The basis you are given may not be orthonormal, so you have to *orthnormalize* it. This process of making the basis orthonormal is the *Gram–Schmidt process*.

Example 12.5. In the plane $\mathbf{R}^2$, every pair of vectors that do not lie on the same line form a basis for the plane. However, some pairs of vectors $\mathbf{u}, \mathbf{v}$ are more special than others - those which lie at a $90°$ angle to each other. Equivalently, it is those pairs $\mathbf{u}, \mathbf{v}$ for which $\text{proj}_{\mathbf{u}}(\mathbf{v}) = \text{proj}_{\mathbf{v}}(\mathbf{u}) = 0$.





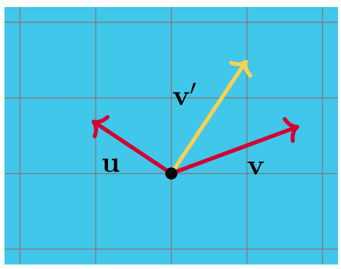
Vectors perpendicular to each other are much easier to deal with, so we try to only work with those. This is the case also for subspaces of vector spaces, for example the plane H defined by $2x+3y-2z = 0$ in $\mathbf{R}^3$. To find the two basis vectors of this plane, we compute a nullspace:

$$H = \text{null}\left(\begin{bmatrix}2 & 3 & -2\end{bmatrix}\right) = \text{null}\left(\begin{bmatrix}1 & \frac{3}{2} & -1\end{bmatrix}\right) = \text{span}\left(\begin{bmatrix}-\frac{3}{2}\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right).$$

These two vectors in the span are not orthogonal to each other, as

$$\underbrace{\begin{bmatrix}-\frac{3}{2}\\1\\0\end{bmatrix}}_{\mathbf{u}} \bullet \underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\mathbf{v}} = -\frac{3}{2} \neq 0.$$

In the plane H:



We would like to make them orthogonal to get a nicer basis. All that we need is to make $\mathbf{v}$ orthogonal to $\mathbf{u}$, and recalling that everything in the orthogonal complement of $\text{span}(\mathbf{u})$ will fulfill this criteria, we simply project $\mathbf{v}$ onto $\text{span}(\mathbf{u})^{\perp}$. Following the formula in Remark 10.5, this new vector is

$$\mathbf{v}' = \text{proj}_{\text{span}(\mathbf{u})^{\perp}}(\mathbf{v}) = (I-P)\mathbf{v} = \mathbf{v} - \frac{\mathbf{v} \bullet \mathbf{u}}{\mathbf{u} \bullet \mathbf{u}}\mathbf{u} = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{-3/2}{9/4+1}\begin{bmatrix}-\frac{3}{2}\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{-6}{13}\begin{bmatrix}-\frac{3}{2}\\1\\0\end{bmatrix} = \begin{bmatrix}1-\frac{9}{13}\\\frac{6}{13}\\1\end{bmatrix} = \begin{bmatrix}\frac{4}{13}\\\frac{6}{13}\\1\end{bmatrix}.$$

The two vectors $\mathbf{u}, \mathbf{v}'$ still span H, but now we have the added benefit of orthogonality:

$$\mathbf{u} \bullet \mathbf{v}' = \begin{bmatrix}-\frac{3}{2}\\1\\0\end{bmatrix} \bullet \begin{bmatrix}\frac{4}{13}\\\frac{6}{13}\\1\end{bmatrix} = -\frac{6}{13} + \frac{6}{13} = 0.$$

Algorithm 5 (The Gram–Schmidt Process): Suppose you have a set $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ of linearly independent vectors. The Gram–Schmidt pocess will first create a set of orthogonal vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n \in V$, and then a set of orthonormal vectors $\mathbf{q}_1, \ldots, \mathbf{q}_n \in V$. They will have all the same span: $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_n) = \text{span}(\mathbf{q}_1, \ldots, \mathbf{q}_n)$.

- Let $\mathbf{w}_1 = \mathbf{v}_1$
- For each $i = 2, \ldots, n$:
  - Let $\mathbf{w}_i = \mathbf{v}_i - \Big(\text{proj}_{\mathbf{w}_{i-1}}(\mathbf{v}_i) + \cdots + \text{proj}_{\mathbf{w}_1}(\mathbf{v}_i)\Big)$.
- The orthonormal set of vectors is $\mathbf{q}_i = \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$.

Inquiry 12.6: Let V be a vector space and $\mathbf{u}, \mathbf{v} \in V$ be linearly independent vectors.

1. What will be the output of the Gram–Schmidt process when it is run on $\mathbf{v}, 2\mathbf{v}, 3\mathbf{v}$?
2. What will be the output of the Gram–Schmidt process when it is run on $\mathbf{v}, \mathbf{u}, \mathbf{v}+\mathbf{u}$?
3. Explain why running the Gram–Schmidt process on the two sets $\mathbf{v}, \mathbf{u}, \mathbf{v}+\mathbf{u}$ and $\mathbf{v}, \mathbf{u}+\mathbf{v}, \mathbf{u}$ in that order will give the same result.

Example 12.7. Consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbf{R}^4$, placed as columns in the matrix

$$\underbrace{\begin{bmatrix}1&2&0&2\\2&0&1&1\\0&2&1&1\\1&0&1&2\end{bmatrix}}_{A} \xrightarrow{RREF} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

These vectors form a basis, but the basis is clearly not orthnormal. If it were, the computations below should give values 1 on the diagonal and 0 everywhere else:

$$\underbrace{\begin{bmatrix}1&2&0&1\\2&0&2&0\\0&1&1&1\\2&1&1&2\end{bmatrix}}_{A^T}\underbrace{\begin{bmatrix}1&2&0&2\\2&0&1&1\\0&2&1&1\\1&0&1&2\end{bmatrix}}_{A} = \begin{bmatrix}6&2&3&4\\2&8&2&4\\3&2&3&4\\4&4&4&10\end{bmatrix} \neq I_4.$$

Exercise 12.2 works through the Gram–Schmidt process on these vectors.

Inquiry 12.8: The three vectors $\mathbf{u} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}1\\-2\\0\end{bmatrix}$ span all of $\mathbf{R}^3$.

- Run the Gram–Schmidt process on $\mathbf{u}, \mathbf{v}, \mathbf{w}$, in that order, and then on the different order $\mathbf{u}, \mathbf{w}, \mathbf{v}$. You may use a computer.
- Why are the results different? Is the span of the resulting vectors different?
- How do you think the two results are related?

If possible, visualize the locations of the vectors on a computer.

12.2 Factorizing and extending

As now is very common, we consider vectors as columns of matrices. Given some vectors as columns in A, and the resulting orthonormal vectors in Q, a natural question arises: How are A and Q related?

Proposition 12.9. There exists a matrix R for which $A = QR$, or $R = Q^T A$, and it is given by

$$R = \begin{bmatrix} - & \mathbf{q}_1 & - \\ - & \mathbf{q}_2 & - \\ - & \mathbf{q}_3 & - \\ - & \mathbf{q}_4 & - \end{bmatrix} \begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1^T\mathbf{v}_1 & \mathbf{q}_1^T\mathbf{v}_2 & \mathbf{q}_1^T\mathbf{v}_3 & \mathbf{q}_1^T\mathbf{v}_4 \\ 0 & \mathbf{q}_2^T\mathbf{v}_2 & \mathbf{q}_2^T\mathbf{v}_3 & \mathbf{q}_2^T\mathbf{v}_4 \\ 0 & 0 & \mathbf{q}_3^T\mathbf{v}_3 & \mathbf{q}_3^T\mathbf{v}_4 \\ 0 & 0 & 0 & \mathbf{q}_4^T\mathbf{v}_4 \end{bmatrix}.$$

The proof of this statement follows immediately by observing that the construction of the $\mathbf{q}_i$ meant that $\mathbf{q}_i \bullet \mathbf{v}_j = 0$ whenever $j < i$. Indeed, we first note that $\mathbf{q}_i \bullet \mathbf{w}_j = 0$ whenever $i \neq j$, since the $\mathbf{q}_i$ point in the same direction as the $\mathbf{w}_i$. So for example,

$$\mathbf{q}_4 \bullet \mathbf{v}_3 = \mathbf{q}_4 \bullet \big(\mathbf{w}_3 + \text{proj}_{\mathbf{w}_1}(\mathbf{v}_3) + \text{proj}_{\mathbf{w}_2}(\mathbf{v}_3)\big) = \underbrace{\mathbf{q}_4 \bullet \mathbf{w}_3}_{0} + \underbrace{\mathbf{q}_4 \bullet \text{proj}_{\mathbf{w}_1}(\mathbf{v}_3)}_{0 \text{ because } \mathbf{q}_4 \bullet \mathbf{w}_1 = 0} + \underbrace{\mathbf{q}_4 \bullet \text{proj}_{\mathbf{w}_2}(\mathbf{v}_3)}_{0 \text{ because } \mathbf{q}_4 \bullet \mathbf{w}_2 = 0} = 0.$$

Remark 12.10. Recall that to find the least squares solution to $A\mathbf{x} = \mathbf{b}$, we projected $\mathbf{b}$ onto col(A) as $\mathbf{p}$. Since

$$A\mathbf{x} = \mathbf{b} = \underbrace{\mathbf{p}}_{\text{in col}(A)} + \underbrace{\mathbf{e}}_{\text{orthogonal to col}(A)}$$

has no solution, but

$$A^T A\mathbf{x} = A^T\mathbf{b} = \underbrace{A^T\mathbf{p}}_{\text{in col}(A^T A)} + \underbrace{A^T\mathbf{e}}_{0}$$

does, least squares was about solving $A^T A\mathbf{x} = A^T\mathbf{b}$. Using the result from Proposition 12.9, this equation becomes

$$\begin{aligned} A^T A\mathbf{x} &= A^T\mathbf{b} \\ (QR)^T(QR)\mathbf{x} &= (QR)^T\mathbf{b} \\ R^T Q^T QR\mathbf{x} &= R^T Q^T\mathbf{b} \\ R^T R\mathbf{x} &= R^T Q^T\mathbf{b} && \text{(since } Q^T Q = I\text{)} \\ R\mathbf{x} &= Q^T\mathbf{b} && \text{(since } R \text{ and } R^T \text{ have inverses)} \\ \mathbf{x} &= R^{-1}Q^T\mathbf{b} && \text{(since } R \text{ has an inverse)} \end{aligned}$$

which requires much less multiplications for a computer to do that $\mathbf{x} = (A^T A)^{-1}A^T\mathbf{b}$.

Inquiry 12.11: Consider the vector space of all functions $[0,1] \to [0,1]$, similar to Exercise 5.3, with the "dot product" defined by $f \bullet g = \int_0^1 f(x)g(x)\,dx$.

- Are the two functions x, x^2 linearly independent? Are they orthogonal?
- Run the Gram–Schmidt process on x, x^2 to get an orthonormal set of functions.
- Changing the space to set of all functions $[0, 2\pi] \to [0, \pi]$, check that $\sin(x), \cos(x)$ are orthogonal.

Remark 12.12. The Gram–Schmidt process is useful for *extending* a basis. That is, given an orthonormal basis for $U \subseteq V$, we can extend the basis to a basis for all of V by simply running the Gram–Schmidt process on the vectors in the given basis, and add as many vectors from V as necessary. For example, given

$$V = \mathbf{R}^4 = \operatorname{span}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) \qquad \text{and} \qquad U = \operatorname{span}\left(\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\2\\2\\1\end{bmatrix}\right) \subseteq V,$$

we can extend the two vectors in the basis of U to a basis of V. Since the given basis vectors of U are orthogonal (but not orthonormal), the first part of Gram–Schmidt process will not affect them. Since $V = \mathbf{R}^4$ is 4-dimensional, we know two facts:

- two vectors are not enough for a basis of V, so $\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\2\\2\\1\end{bmatrix}$ is too small to be a basis, and
- six vectors are too many for a basis of V, so $\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\2\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ is too big to be a basis.

To find an orthonormal basis of V that contains the two basis vectors of U, simply run the Gram–Schmidt process on all six vectors, beginning with the two from the basis of U.

12.3 Exercises

Exercise 12.1. Check that the columns of the 2×2 rotation matrix (introduced in Lecture 9.1) and of the 3×3 permutation matrices (introduced in Lecture **??**) are all orthogonal. Are they orthonormal?

Exercise 12.2. Apply the Gram-Schmidt process to the vectors $\begin{bmatrix}1\\2\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\\2\end{bmatrix}$.

Exercise 12.3. Consider the 2-dimensional subspace $H \subseteq \mathbf{R}^4$ defined by

$$H = \left\{ (x, y, z, w) \in \mathbf{R}^4 \; : \; \begin{array}{rcl} 2x + 3y - w & = & 0, \\ y - z + 2w & = & 0. \end{array} \right\}$$

1. Express H as a span of two vectors.
2. Apply the Gram–Schmidt process to the two vectors from above to get H as a span of two orthonormal vectors.
3. The space $\mathbf{R}^4$ has the xy-plane as a 2-dimensional subspace, with basis $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$. Give the change of basis matrix from the two vectors in part 2. to these two vectors.

Lecture 13: Generalized distances

Chapter IV.10 in Strang's "Learning from Data"

- Fact 1: The inner product generalizes the concept a distance for other spaces.
- Fact 2: (Relative) positions of points cna be recovered knowing just the distances between them.

- Skill 1: Compute the length of, angle between, and projections of vectors in arbitrary inner product spaces.
- Skill 2: Construct the position matrix knowing just the distance matrix.

We now take a small detour from Strang's *Linear Algebra* and work with the material from Strang's *Learning from Data*. The topic follows the topics of the previous lectures, expanding on the idea of orthogonality and unit length in different vector spaces.

13.1 Functions on spaces

Definition 13.1: Let V be a vector space. An *inner product* on V is a function $\langle \cdot\,, \cdot \rangle : V^2 \to \mathbf{R}$ such that for all $\mathbf{v}, \mathbf{u}, \mathbf{w} \in V$ and all $c \in \mathbf{R}$,

- (positive definite) $\langle \mathbf{v}, \mathbf{v} \rangle \geqslant 0$ with $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$
- (symmetric) $\langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$
- (multiplicative) $\langle c\mathbf{v}, \mathbf{u} \rangle = c\langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{v}, c\mathbf{u} \rangle$
- (bilinear) $\langle \mathbf{v} + \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

A vector space V that has an inner product is called an *inner product space*. Given any two vectors $\mathbf{u}, \mathbf{v}$ in an inner product space V,

- they are *orthogonal* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$,
- the *angle* $\theta \in [0, 2\pi)$ between them is given by $\cos(\theta) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

Recall that the only required operations for a vector space were scalar multiplication and vector addition (a dot product was not required).

We have already seen an example of the inner product in the *dot product* of two vectors. Just like there, every inner product has a notion of distance associated to it: the *norm*, or *length*, of $\mathbf{v}$ in an inner product space V is

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\mathbf{v} \bullet \mathbf{v}}.$$

Example 13.2. There are many examples of inner product spaces besides $\mathbf{R}^n$ with the dot product.

- The space $\mathcal{M}_{m\times n}$ of all $m \times n$ matrices over $\mathbf{R}$ is an inner product space when using $\langle A, B \rangle := \operatorname{trace}(A^T B)$. The *trace* is the sum of the entries on the diagonal.
- The space $C[0, 1]$ of all continuous functions with domain $[0, 1]$ and inner product

$$\langle f, g \rangle := \int_0^1 f(x)g(x)\, dx$$

  is an inner product space. Adjusting the domain to any interval $[a, b] \subseteq \mathbf{R}$ still makes this an inner product space.

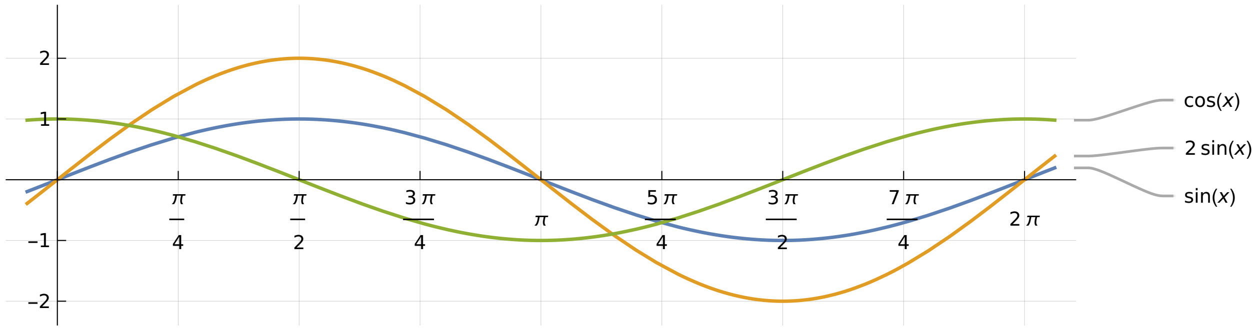
Inquiry 13.3: This inquiry is about the properties of an inner product space V given in Definition 13.1. Using them, show that:

- $\langle 0, \mathbf{v}\rangle = \langle \mathbf{v}, 0\rangle$ for all $\mathbf{v} \in V$
- the only vector in V that is orthogonal to itself is 0
- the *parallelogram equality* holds: $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$

Theorem 13.4. *The inner product* $\langle\ \cdot\ ,\ \cdot\ \rangle$ *in any inner product space* $V \ni \boldsymbol{v}, \boldsymbol{w}$ *satisfies:*

- *the Cauchy–Schwarz inequaity:* $|\langle \boldsymbol{v}, \boldsymbol{w}\rangle| \leqslant \|\boldsymbol{v}\|\|\boldsymbol{w}\|$ *with equality iff* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *are linearly dependent*
- *the triangle inequality:* $\|\boldsymbol{v}+\boldsymbol{w}\| \leqslant \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$

Example 13.5. Using the first point of Theorem 13.4, we can show that the functions $\sin(x)$ and $\cos(x)$ are linearly independent in $C[0, 2\pi]$, and that $\sin(x)$ and $2\sin(x)$ are linearly dependent.



We find that

$$\begin{aligned}
\langle \sin(x), \cos(x)\rangle &= \int_0^{2\pi} \sin(x)\cos(x)\ dx = \int_0^{2\pi} \frac{\sin(2x)}{2}\ dx = \frac{-\cos(4\pi)}{4} - \frac{-\cos(0)}{4} = 0,\\
\|\sin(x)\|^2 &= \int_0^{2\pi} \sin^2(x)\ dx = \int_0^{2\pi} \frac{1-\cos(2x)}{2}\ dx = \pi - \left(\frac{\sin(4\pi)}{4} - \frac{\sin(0)}{4}\right) = \pi,\\
\|\cos(x)\|^2 &= \int_0^{2\pi} \cos^2(x)\ dx = \int_0^{2\pi} \frac{\cos(2x)+1}{2}\ dx = \left(\frac{\sin(4\pi)}{4} - \frac{\sin(0)}{4}\right) + \pi = \pi.
\end{aligned}$$

Since $0 \neq \sqrt{\pi}\cdot\sqrt{\pi} = \pi$, the functions $\sin(x)$ and $\cos(x)$ are linearly independent, but since $2\pi = \sqrt{\pi}\cdot\sqrt{4\pi}$, the functions $\sin(x)$ and $2\sin(x)$ are linearly dependent. Also note that the positive definite property of the inner product is satisfied.

The notions of angle between vectors, orthogonality, unit length, all apply to inner product spaces in the same way they applied to $\mathbf{R}^n$ with the dot product.

Example 13.6. The angle between the matrices $A = \left[\begin{smallmatrix} 4 & 1\\ -1 & 0\\ 7 & 2\end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 2\\ 3 & -1\\ 2 & 0\end{smallmatrix}\right]$ is

$$\begin{aligned}
\cos^{-1}\left(\frac{\operatorname{trace}(A^TB)}{\operatorname{trace}(A^TA)\operatorname{trace}(B^TB)}\right) &= \cos^{-1}\left(\frac{\operatorname{trace}\left(\left[\begin{smallmatrix} 4 & -1 & 7\\ 1 & 0 & 2\end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 2\\ 3 & -1\\ 2 & 0\end{smallmatrix}\right]\right)}{\operatorname{trace}\left(\left[\begin{smallmatrix} 4 & -1 & 7\\ 1 & 0 & 2\end{smallmatrix}\right]\left[\begin{smallmatrix} 4 & 1\\ -1 & 0\\ 7 & 2\end{smallmatrix}\right]\right)\operatorname{trace}\left(\left[\begin{smallmatrix} 0 & 3 & 2\\ 2 & -1 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 2\\ 3 & -1\\ 2 & 0\end{smallmatrix}\right]\right)}\right)\\
&= \cos^{-1}\left(\frac{\operatorname{trace}\left(\left[\begin{smallmatrix} 9 & 9\\ 4 & 2\end{smallmatrix}\right]\right)}{\operatorname{trace}\left(\left[\begin{smallmatrix} 66 & 18\\ 18 & 5\end{smallmatrix}\right]\right)\operatorname{trace}\left(\left[\begin{smallmatrix} 13 & -3\\ -3 & 5\end{smallmatrix}\right]\right)}\right)\\
&= \cos^{-1}\left(\frac{11}{1278}\right)\\
&\approx 89.51^\circ
\end{aligned}$$

Remark 13.7. The Gram–Schmidt process in Lecture 12 was done on vectors using the usual norm in $\mathbf{R}^n$. By obeserving that the projection operation can be given in terms of inner product, the Gram–Schmidt process can be applied to any inner product space:

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{v}^T\mathbf{u}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\langle \mathbf{v},\mathbf{u}\rangle}{\langle \mathbf{v},\mathbf{v}\rangle}\mathbf{v}.$$

Inquiry 13.8: Consider the inner product spaces $C[a,b]$ and $\mathcal{M}_{2\times 2}$.

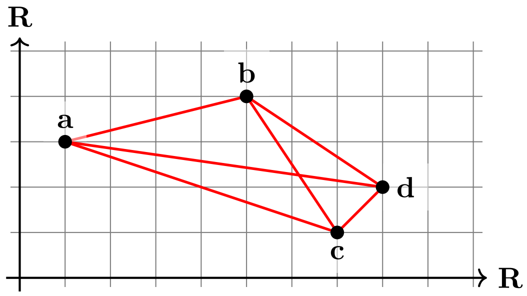
- What is the angle between $x+1$ and x^2+1 in $C[0,1]$?
- Compute the projection of $\cos(x)$ onto $\sin(x)$ in $C[0,2\pi]$.
- Compute the projection of the rotation matrix R_θ onto $\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$. For what angles θ are these matrices orthogonal to each other?

13.2 Distance matrices

Recall the points from Exercise 11.9 in Lecture 12, which were used in the motivating least squares example. If the points were located elsewhere but their *relative* position to each other was the same, we can still solve the least squares problem, up to some x-shift and y-shift. This situation has two advantages:

- *only requires relative information*: measurements only need to be made among the data, not between data and something else (like a reference point - the origin)
- *allows for spaces that are not* $\mathbf{R}^n$: on the sphere, on a grid, with barriers, etc

Example 13.9. Consider the distances among the four points, slightly adapted from Exercise 11.9.



$$\begin{aligned}\mathbf{a} &= (1,3)\\ \mathbf{b} &= (5,4)\\ \mathbf{c} &= (7,1)\\ \mathbf{d} &= (8,2)\end{aligned} \qquad X = \begin{bmatrix}1 & 5 & 7 & 8\\ 3 & 4 & 1 & 2\end{bmatrix}$$

The matrix X is called the *position matrix*. We can easily compute the symmetric *distance matrix*

$$D = \begin{bmatrix}\|\mathbf{a}-\mathbf{a}\|^2 & \|\mathbf{a}-\mathbf{b}\|^2 & \|\mathbf{a}-\mathbf{c}\|^2 & \|\mathbf{a}-\mathbf{d}\|^2\\ \|\mathbf{b}-\mathbf{a}\|^2 & \|\mathbf{b}-\mathbf{b}\|^2 & \|\mathbf{b}-\mathbf{c}\|^2 & \|\mathbf{b}-\mathbf{d}\|^2\\ \|\mathbf{c}-\mathbf{a}\|^2 & \|\mathbf{c}-\mathbf{b}\|^2 & \|\mathbf{c}-\mathbf{c}\|^2 & \|\mathbf{c}-\mathbf{d}\|^2\\ \|\mathbf{d}-\mathbf{a}\|^2 & \|\mathbf{d}-\mathbf{b}\|^2 & \|\mathbf{d}-\mathbf{c}\|^2 & \|\mathbf{d}-\mathbf{d}\|^2\end{bmatrix} = \begin{bmatrix}0 & 17 & 40 & 50\\ 17 & 0 & 13 & 13\\ 40 & 13 & 0 & 2\\ 50 & 13 & 2 & 0\end{bmatrix},$$

which contains the squares of the distnce among the points. The relationship between X and D is not so clear, however.

Proposition 13.10. Let $D \in \mathcal{M}_{k\times k}$ be the matrix containg squares of distances among k points $\mathbf{v}_1,\ldots,\mathbf{v}_k \in \mathbf{R}^n$. The relationship between D and the position matrix X is given by

$$X^TX = \frac{1}{2}\left(\mathbf{s}\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix} + \begin{bmatrix}1\\1\\ \vdots\\ 1\end{bmatrix}\mathbf{s}^T - D\right),$$

where the position vector $\mathbf{s}^T = \left[\, \|\mathbf{v}_1 - \mathbf{v}_1\|^2 \;\; \|\mathbf{v}_1 - \mathbf{v}_2\|^2 \;\cdots\; \|\mathbf{v}_1 - \mathbf{v}_2\|^2 \,\right]$ is the first row of the matrix D.

Example 13.11. Continuing Example 13.9, we fix one of the points as a reference point. Without loss of generality, we simply say

$$\mathbf{a} = 0.$$

That is, we subtract $\mathbf{a}$ from all the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ to get new ones (which we call the same). Any other vector $\mathbf{b}, \mathbf{c}, \mathbf{d}$ could have been chose. Now the first line of D becomes (squares of) the lengths $\|\cdot\|$ of all the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, and the lengths also appear on the diagonal of $X^T X$:

$$D = \begin{bmatrix} 0 & \|\mathbf{b}\|^2 & \|\mathbf{c}\|^2 & \|\mathbf{d}\|^2 \\ \|\mathbf{b}\|^2 & 0 & \|\mathbf{b}-\mathbf{c}\|^2 & \|\mathbf{b}-\mathbf{d}\|^2 \\ \|\mathbf{c}\|^2 & \|\mathbf{c}-\mathbf{b}\|^2 & 0 & \|\mathbf{c}-\mathbf{d}\|^2 \\ \|\mathbf{d}\|^2 & \|\mathbf{d}-\mathbf{b}\|^2 & \|\mathbf{d}-\mathbf{c}\|^2 & 0 \end{bmatrix}, \qquad X^T X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbf{b}\bullet\mathbf{b} & \mathbf{b}\bullet\mathbf{c} & \mathbf{b}\bullet\mathbf{d} \\ 0 & \mathbf{c}\bullet\mathbf{b} & \mathbf{c}\bullet\mathbf{c} & \mathbf{c}\bullet\mathbf{d} \\ 0 & \mathbf{d}\bullet\mathbf{b} & \mathbf{d}\bullet\mathbf{c} & \mathbf{d}\bullet\mathbf{d} \end{bmatrix}.$$

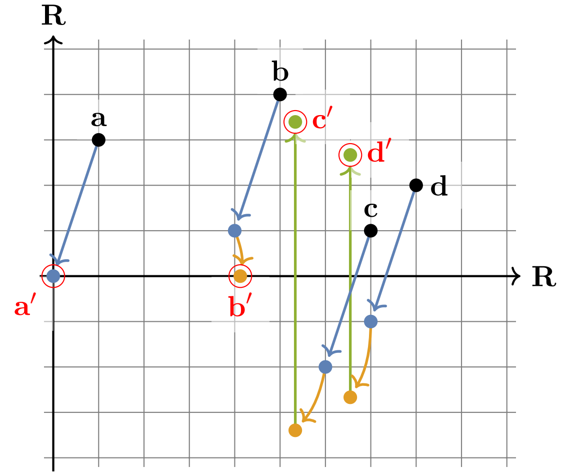
Applying the result of Proposition 13.10, we construct the position vector $\mathbf{s}^T = \left[\, 0 \;\; \|\mathbf{b}\|^2 \;\; \|\mathbf{c}\|^2 \;\; \|\mathbf{d}\|^2 \,\right] = \left[\, 0 \;\; 17 \;\; 40 \;\; 50 \,\right]$, and compute

$$\begin{aligned}
X^T X &= \frac{1}{2}\left(\mathbf{s}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\mathbf{s}^T - D \right) \\
&= \frac{1}{2}\left(\begin{bmatrix}0\\17\\40\\50\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}0 & 17 & 40 & 40\end{bmatrix} - \begin{bmatrix}0 & 17 & 40 & 50\\17 & 0 & 13 & 13\\40 & 13 & 0 & 2\\50 & 13 & 2 & 0\end{bmatrix} \right) \\
&= \frac{1}{2}\left(\begin{bmatrix}0 & 0 & 0 & 0\\17 & 17 & 17 & 17\\40 & 40 & 40 & 40\\50 & 50 & 50 & 50\end{bmatrix} + \begin{bmatrix}0 & 17 & 40 & 50\\0 & 17 & 40 & 50\\0 & 17 & 40 & 50\\0 & 17 & 40 & 50\end{bmatrix} - \begin{bmatrix}0 & 17 & 40 & 50\\17 & 0 & 13 & 13\\40 & 13 & 0 & 2\\50 & 13 & 2 & 0\end{bmatrix} \right) \\
&= \begin{bmatrix}0 & 0 & 0 & 0\\0 & 17 & 22 & 27\\0 & 22 & 40 & 44\\0 & 27 & 44 & 50\end{bmatrix}.
\end{aligned}$$

Since $X^T X$ is symmetric, doing row reduction to get the LDU-decomposition of $X^T X$ will produce symmetric matrices, that is, $LDU = (L\sqrt{D})(\sqrt{D}U)$, with $L = U^T$. This will give X, up to a shift and potentially a rotation and a reflection. Note that the matrix "D" here isthe diagonal matrix from the LDU-decomposition, and is different from the distance matrix "D" used above.
For this example, we have

$$\begin{bmatrix}0 & 0 & 0 & 0\\0 & 17 & 22 & 27\\0 & 22 & 40 & 44\\0 & 27 & 44 & 50\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & \frac{22}{17} & 1 & 0\\0 & \frac{27}{17} & \frac{11}{14} & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}0 & 0 & 0 & 0\\0 & 17 & 22 & 27\\0 & 0 & \frac{196}{17} & \frac{154}{17}\\0 & 0 & 0 & 0\end{bmatrix}}_{U} = \underbrace{\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & \frac{22}{17} & 1 & 0\\0 & \frac{27}{17} & \frac{11}{14} & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1 & 0 & 0 & 0\\0 & 17 & 0 & 0\\0 & 0 & \frac{196}{17} & 0\\0 & 0 & 0 & 1\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}0 & 0 & 0 & 0\\0 & 1 & \frac{22}{17} & \frac{27}{17}\\0 & 0 & 1 & \frac{11}{14}\\0 & 0 & 0 & 10\end{bmatrix}}_{U},$$

where we note that L is the transpose of U if we ignore the zero rows of U. From this we recover the points $\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}'$ as the non zero rows of $\sqrt{D}U$.



$$\begin{aligned} \mathbf{a}' &= (0,0) \\ \mathbf{b}' &= (\sqrt{17}, 0) \\ \mathbf{c}' &= (\tfrac{22}{\sqrt{17}}, \tfrac{14}{\sqrt{17}}) \\ \mathbf{d}' &= (\tfrac{27}{\sqrt{17}}, \tfrac{11}{\sqrt{17}}) \end{aligned}$$

$$X' = \begin{bmatrix} 0 & \sqrt{17} & \frac{22}{\sqrt{17}} & \frac{27}{\sqrt{17}} \\ 0 & 0 & \frac{14}{\sqrt{17}} & \frac{11}{\sqrt{17}} \end{bmatrix}$$

Inquiry 13.12: Consider vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in $\mathbf{R}^2$. If we know only the matrix D of distances between them, the recovery method presented in Example 13.9 computes the positions of $\mathbf{a} - \mathbf{a}, \mathbf{b} - \mathbf{a}, \mathbf{c} - \mathbf{a}, \mathbf{d} - \mathbf{a}$.

- Suppose instead $\mathbf{b}$ was subtracted from all the vectors. What is the relationship between the vectors recovered in this way to those recovered by subtracting $\mathbf{a}$?
- Suppose you have 4 new vectors, which are just $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ rotated by 90° clockwise. After applying the recovery method to get X, how are the recovered vectors related to the vectors recovered by the first method?

Remark 13.13. If instead we have a set of vectors $\mathbf{v}_i, \ldots, \mathbf{v}_k$, then the distance matrix would be defined as $D_{ij} = \|\mathbf{v}_i - \mathbf{v}_j\|$. Note that this means the distance matrix is always symmetric and has a zero diagonal.

Example 13.14. If D is simply symmetric and has a zero diagonal, there is no guarantee that is represents distance among points in a space like $\mathbf{R}^n$, or even any inner product space. Consider the distance matrix

$$D = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 3 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & 3 & 3 & 0 \end{bmatrix},$$

coming from four points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$. As in the previous example, we let the first point $\mathbf{a} = 0$, so that we get $\|\mathbf{b}\| = \|\mathbf{c}\| = \|\mathbf{d}\| = 1$. We also see that

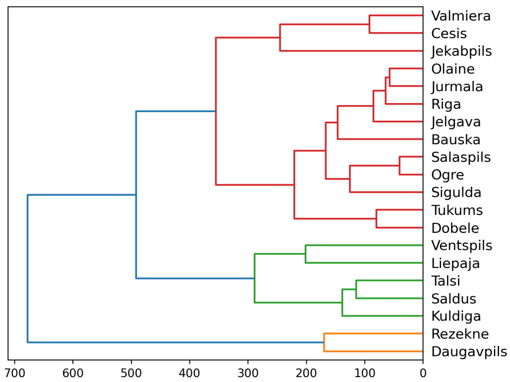
$$\begin{aligned} 3^2 &= \|\mathbf{b} - \mathbf{c}\|^2 \\ &= \langle \mathbf{b} - \mathbf{c}, \mathbf{b} - \mathbf{c} \rangle \\ &= \langle \mathbf{b}, \mathbf{b} - \mathbf{c} \rangle - \langle \mathbf{c}, \mathbf{b} - \mathbf{c} \rangle \\ &= \langle \mathbf{b}, \mathbf{b} \rangle - \langle \mathbf{b}, \mathbf{c} \rangle - \langle \mathbf{c}, \mathbf{b} \rangle + \langle \mathbf{c}, \mathbf{c} \rangle \\ &= \|\mathbf{b}\|^2 - 2\langle \mathbf{b}, \mathbf{c} \rangle + \|\mathbf{c}\|^2 \\ &= 1 - 2\langle \mathbf{b}, \mathbf{c} \rangle + 1. \end{aligned}$$

Rearranging, we conclude that $\langle \mathbf{b}, \mathbf{c} \rangle = -7/2$, which contradicts the fact that the inner product must be positive definite. Hence D can not be a distance matrix of points from an inner product space.

Distance matrices can highlight *clustering* among the data. That is, given a distance matrix, we can "connect" points that lie close to each other and so discover which groups of points are close to each other.

Example 13.15. Consider the distances between the 20 largest cities in Latvia, in kilometers. As a distance matrix, it is difficult to get information from it, but we can group cities by distance into clusters. This could be useful, for example, in trying to decide where to build a factory or distribution center.

Bauska	0	119	152	62	103	37	68	149	196	53	46	194	62	54	111	91	136	89	145	191
Cesis	119	0	174	137	96	115	93	200	271	66	97	155	79	69	180	30	159	130	27	220
Daugavpils	152	174	0	214	79	186	203	301	344	156	185	89	187	166	263	170	281	235	194	340
Dobele	62	137	214	0	158	29	47	87	141	81	41	249	59	72	51	108	78	34	158	129
Jekabpils	103	96	79	158	0	129	134	242	297	85	122	91	117	96	208	92	215	171	116	276
Jelgava	37	115	186	29	129	0	38	115	170	54	19	220	40	47	80	86	98	51	139	154
Jurmala	68	93	203	47	134	38	0	111	179	49	22	221	18	39	88	65	80	39	112	142
Kuldiga	149	200	301	87	242	115	111	0	79	159	120	331	129	149	42	175	48	72	214	48
Liepaja	196	271	344	141	297	170	179	79	0	221	181	389	195	212	92	244	127	141	288	98
Ogre	53	66	156	81	85	54	49	159	221	0	40	172	32	11	130	38	129	87	91	191
Olaine	46	97	185	41	122	19	22	120	181	40	0	211	21	31	90	68	96	50	120	156
Rezekne	194	155	89	249	91	220	221	331	389	172	211	0	203	183	299	166	300	259	163	362
Riga	62	79	187	59	117	40	18	129	195	32	21	203	0	21	103	50	98	57	100	159
Salaspils	54	69	166	72	96	47	39	149	212	11	31	183	21	0	120	40	119	77	94	180
Saldus	111	180	263	51	208	80	88	42	92	130	90	299	103	120	0	153	61	52	198	89
Sigulda	91	30	170	108	92	86	65	175	244	38	68	166	50	40	153	0	137	104	54	200
Talsi	136	159	281	78	215	98	80	48	127	129	96	300	98	119	61	137	0	47	170	62
Tukums	89	130	235	34	171	51	39	72	141	87	50	259	57	77	52	104	47	0	147	105
Valmiera	145	27	194	158	116	139	112	214	288	91	120	163	100	94	198	54	170	147	0	229
Ventspils	191	220	340	129	276	154	142	48	98	191	156	362	159	180	89	200	62	105	229	0



To get the *dendrogram* above, each city begins in its own cluster. The two closest cities are connected to create one cluster of 2 cities (Ogre and Salaspils). Create larger clusters by measuring the distance between every pair of clusters c_i and c_j, with distance defined to be

$$\text{(distance between } c_i \text{ and } c_j\text{)} \ = \frac{1}{|c_i||c_j|} \sum_{\mathbf{v}_i \in c_i} \sum_{\mathbf{v}_j \in c_j} \|\mathbf{v}_i - \mathbf{v}_j\|.$$

For clusters of size 1, note that $|c_i| = |c_j| = 1$, and the distance reduces to the usual distance. This is the *average* method of drawing a dendrogram. In the diagram above, the last 3 clusters to be joined are colored differently, but any number can be chosen here.

Inquiry 13.16: This question is about coding in Python.

- Generate a collection of 333 random points in $\mathbf{R}^2$, with:
  - 300 of them randomly selected from $[0,1] \times [0,1]$,
  - 30 of them randomly selected from $[5,6] \times [5,6]$,
  - 3 of them randomly selected from $[1,2] \times [8,9]$.
- Construct the 333×333 distance matrix between them.
- Construct the dendrogram from this matrix. Does it reflect the clusters as you created them?

13.3 Exercises

Exercise 13.1. For each of the following "definitions", show that each cannot be an inner product.

- For $A, B \in \mathcal{M}_{n\times n}$, let $\langle A, B\rangle = \text{trace}(A + B)$
- For $f, g \in C[0,1]$, let $\langle f, g\rangle = \left|\frac{df}{dx}\frac{dg}{dx}\right|$
- For $a, b \in \mathbf{R}$, let $\langle a, b\rangle = a^2 + b^2$

Exercise 13.2. Check the conditions for the space of $m \times n$ matrices over $\mathbf{R}$ from Example 13.2 being an inner product space. What is the distance between $\left[\begin{smallmatrix}1&2&3\\4&5&6\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0&1&2\\3&4&5\end{smallmatrix}\right]$?

Exercise 13.3. Consider the following three matrices in $\mathcal{M}_{2\times 2}$:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & -3 \\ 3 & 2 \end{bmatrix}.$$

Using the Gram–Schmidt process to find an orthonormal basis for span$\{A, B, C\}$. Use the inner product on matrices given in Example 13.2.

Exercise 13.4. Let $P(\mathbf{R})$ be the vector space of all polynomials $\mathbf{R} \to \mathbf{R}$, with scalar multiplication and polynomial addition defined as you would expect. You may assume that the following is an inner product on $P(\mathbf{R})$:

$$\langle p(x), q(x) \rangle = \int_0^\infty p(x)q(x)e^{-x}\, dx.$$

1. Check that $p(x) = 2x - 1$ and $q(x) = x + 3$ are not orthogonal to each other.
2. Using the Gram–Schmidt process on $p(x)$ and $q(x)$ as in part 1., find a polynomial $r(x) \in P(\mathbf{R})$ that is orthogonal to $p(x)$. Give your answer as $r(x) = ax + b$, for $a, b \in \mathbf{Z}$.

Exercise 13.5. Given the distance D matrix below, construct the dendrogram using the same average distance method as in Example 13.15. After every step, give the new distance matrix, which measures the distances among the clusters.

$$D = \begin{bmatrix} 0 & 12 & 10 & 13 & 2 & 11 \\ 12 & 0 & 3 & 9 & 13 & 8 \\ 10 & 3 & 0 & 6 & 14 & 5 \\ 13 & 9 & 6 & 0 & 15 & 1 \\ 2 & 13 & 14 & 15 & 0 & 7 \\ 11 & 8 & 5 & 1 & 7 & 0 \end{bmatrix}$$

Part III

Eigensystems

Lecture 14: Defining the determinant

Chapters 5.1,5.2 in Strang's "Linear Algebra"

- Fact 1: The determinant may be computed either recursively or combinatorially, only for a square matrix.
- Fact 2: The determinant is related to the pivots and invertibility of a matrix.

- Skill 1: Use both the recursive and combinatorial definitions to compute the determinant.
- Skill 2: Use the definitions of the determinant to show properties of the determinant.

We now begin a new part of this course, on everything to do with *eigenvectors* and *eigenvalues*. The first step is the *determinant* of a matrix, which is a rough estimate of the eigenvalues of the matrix. In fact, the detemrinant is the product of all the eigenvalues.

14.1 The recursive definition

The *determinant* is a function $\det\colon \mathcal{M}_{n\times n} \to \mathbf{R}$, and denoted as either $\det(A)$ or with vertical bars $|A|$. Before we get to definitions and new ideas, we mconsider some concepts you have already seen, in this and previous courses.

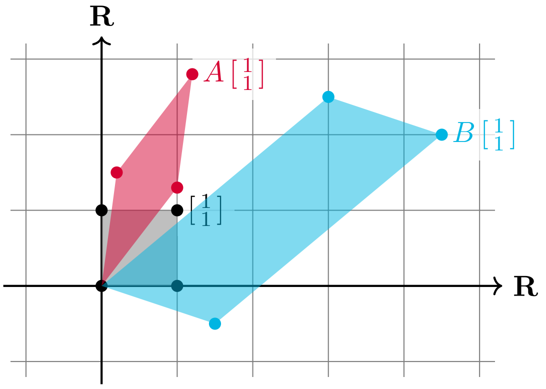
Example 14.1. The determinant is a commonly found number, often associated with *invertibility*.

- (Definition 14.4) The determinant of a 1×1 matrix $\left[\begin{smallmatrix} a \end{smallmatrix}\right]$ is a. The matrix is not invertible is $a = 0$.
- (Definition 14.4) The determinant of a 2×2 matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is $ad-bc$. The matrix is not invertible if $ad - bc = 0$.
- (to be proved later) The determinant is the product of the pivots, up to a sign change.
- (to be proved later) The determinant is zero if and only if the matrix is not invertible.

Definition 14.2: Let $n \in \mathbf{N}$. The *unit n-cube* in $\mathbf{R}^n$ is the set of points $(x_1, \dots, x_n)$ with $0 \leqslant x_i \leqslant 1$ for all i. The unit n-cube has *n-dimensional volume*, or simply *n-volume*, equal to 1. The n-volume of any other shape in $\mathbf{R}^n$ is given by the number of (fractions of) unit n-cubes in the shape.

This way to define n-dimensional volume is a rough estimate of the more accurate way, which would be to take an n-fold integral.

Example 14.3. Consider $A \in \mathcal{M}_{n\times n}$ as a function $\mathbf{R}^n \to \mathbf{R}^n$. The absolute value of the determinant of A is the n-dimensional volume of the shape that the unit n-cube becomes, after multiplying each of its corners by A.



$$A = \begin{bmatrix} 1 & \frac{1}{5} \\ \frac{13}{10} & \frac{3}{2} \end{bmatrix} \qquad \det(A) = 1.24$$

$$B = \begin{bmatrix} \frac{3}{2} & 3 \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} \qquad \det(B) = 5.25$$

The red and blue images are called *parallelograms*. In general, the image of the corners of the unit n-cube, when multiplied by an $n \times n$ matrix, is called a *parallelotope*.

Outr first definition of the determinant is a recursive definition, which justiofies the first two examples in Example 14.1.

Definition 14.4: Let $A \in \mathcal{M}_{n\times n}$. The *determinant* $\det(A)$ of A is:

- if $n = 1$, then $\det(A) = A_{11}$
- if $n \geqslant 2$, then $\det(A) = \displaystyle\sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(A^{ij})$, for any $i \in \{1, \dots, n\}$

The matrix A^{ij} is the $(n-1) \times (n-1)$ *submatrix* of A produced when the ith row and jth column are removed. In this setting,

- the number $\det(A^{ij})$ is called the ij-*minor* of A,
- the number $(-1)^{i+j} \det(A^{ij})$ is called the ij-*cofactor* of A.

The $n \times n$ matrix with ij-entry the ij-cofactor is called the *cofactor matrix* $\operatorname{cofac}(A)$ of A.

Example 14.5. Following Definition 14.4, we compute the determinant of a matrix A, using $i = 1$:

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} 0 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 5 & -2 \end{bmatrix}\right) \\
&= \begin{vmatrix} 0 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 5 & -2 \end{vmatrix} \\
&= (-1)^{1+1} 0 \begin{vmatrix} -1 & 2 \\ 5 & -2 \end{vmatrix} + (-1)^{1+2} 3 \begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} + (-1)^{1+3} 4 \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} \\
&= 0 - 3(-4-2) + 4(10+1) \\
&= 18 + 44 \\
&= 62.
\end{aligned}$$

We would have gotten the same result with $i = 2$ or $i = 3$.

Inquiry 14.6: Let $A \in \mathcal{M}_{n\times n}$. Using Definition 14.4, show that the following statements are true, for any $n \in \mathbf{N}$.

- The determinant of the $n \times n$ identity matrix is 1. That is, $\det(I_n) = 1$.
- The determinant of an upper (or lower) triangular matrix is the product of the diagonal entries.

Hint: use induction for both!

Next we describe some general properties of the determinant.

Proposition 14.7. Let $A \in \mathcal{M}_{n\times n}$. As a function of the rows of A, the determinant is:

- *multilinear*, that is, $\det(\mathbf{r}_1 \ldots, c\mathbf{a}+\mathbf{b}, \ldots, \mathbf{r}_n) = c\det(\mathbf{r}_1, \ldots, \mathbf{a}, \ldots, \mathbf{r}_n) + \det(\mathbf{b}, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_n)$
- *alternating*, that is, $\det(\mathbf{r}_1 \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_j, \ldots, \mathbf{r}_n) = -\det(\mathbf{r}_1 \ldots, \mathbf{r}_j, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_n)$

Proof. The first point follows by induction on n, and by using the recursive definition (Definition 14.4) to expand along row i. The statement is immediately true for a 1×1 matrix. For the inductive step, notice that

$$\begin{aligned}\det A &= \det(\mathbf{r}_1, \ldots, c\mathbf{a}+\mathbf{b}, \ldots, \mathbf{r}_n)\\ &= \sum_{j=1}^{n}(-1)^{i+j}(c\mathbf{a}+\mathbf{b})_j \det(A^{ij})\\ &= c\left(\sum_{j=1}^{n}(-1)^{i+j}(\mathbf{a})_j \det(A^{ij})\right) + \left(\sum_{j=1}^{n}(-1)^{i+j}(\mathbf{b})_j \det(A^{ij})\right),\end{aligned}$$

and A^{ij} is the same in both cases.

The second point follows by using the combinatorial definition of the determinant (Definition 14.13). Fix two different indices $i, j \in \{1, 2, \ldots, n\}$. For every permutation σ on a set of size n, let σ' be the permutation given by

$$\sigma'(k) = \begin{cases} \sigma(k) & k \neq i, j, \\ \sigma(j) & k = i, \\ \sigma(i) & k = j. \end{cases}$$

That is, σ' is the same as σ, except it swaps the images of i and j. Note that $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$, since σ' has one row swap that σ does not have. Now suppose that for a matrix A, the matrix A' is the same, except with rows i and j swapped. Then

$$\begin{aligned}\det(A') &= \sum_{\text{permutations } \sigma} \operatorname{sgn}(\sigma) A'_{1\sigma(1)} A'_{2\sigma(2)} \cdots A'_{i\sigma(i)} \cdots A'_{j\sigma(j)} \cdots A'_{n\sigma(n)} && \text{(definition of det)}\\ &= \sum_{\text{permutations } \sigma} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A'_{i\sigma(i)} \cdots A'_{j\sigma(j)} \cdots A_{n\sigma(n)} && \text{(definition of } A')\\ &= \sum_{\text{permutations } \sigma} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{i\sigma(j)} \cdots A_{j\sigma(i)} \cdots A_{n\sigma(n)} && (A_i = A'_j \text{ and } A_j = A'_i)\\ &= \sum_{\text{permutations } \sigma'} \operatorname{sgn}(\sigma) A_{1\sigma'(1)} A_{2\sigma'(2)} \cdots A_{i\sigma'(i)} \cdots A_{j\sigma'(j)} \cdots A_{n\sigma'(n)} && \text{(definition of } \sigma')\\ &= -\sum_{\text{permutations } \sigma'} \operatorname{sgn}(\sigma') A_{1\sigma'(1)} A_{2\sigma'(2)} \cdots A_{i\sigma'(i)} \cdots A_{j\sigma'(j)} \cdots A_{n\sigma'(n)} && \text{(property of } \sigma')\\ &= -\det(A). && \text{(definition of det)}\end{aligned}$$

□

Example 14.8. Consider the following example of multilinearity (on the left) and the alternating property (on the right):

$$\begin{aligned} -46 &= 11 \cdot (-2) - 8 \cdot 3 \\ &= \begin{vmatrix} 11 & 8 \\ 3 & -2 \end{vmatrix} \\ &= \begin{vmatrix} 6+5 & 9-1 \\ 3 & -2 \end{vmatrix} \\ &= 3 \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 3 & -2 \end{vmatrix} \\ &= 3\,(2 \cdot (-2) - 3 \cdot 3) + (5 \cdot (-2) - (-1) \cdot 3) \\ &= -39 - 7 \\ &= -46. \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} -5 & 6 \\ -5 & 6 \end{vmatrix} &= (-5) \cdot 6 - (-5) \cdot 6 \\ &= -30 + 30 \\ &= 0 \end{aligned}$$

Inquiry 14.9: There are two immediate consequences of Proposition 14.7. Show why they are both true, in general for an $n \times n$ matrix.

- A matrix with a zero row has determinant zero.
- A matrix with two equal rows has determinant zero.

Hint: consider the determinant as a function of the rows, as in the proposition.

14.2 A combinatorial definition

We now consider the determinant in a combinatorial context, that is, as it relates to all permutations of the rows and columns of a matrix.

Definition 14.10: Let $S = (a_1, \ldots, a_n)$ be an ordered set. A *permutation* of S is equivalently

- a bijective function $\sigma : (1, \ldots, n) \to (1, \ldots n)$, or
- a rearrangement of the elements of S in a different order.

A *transposition* is a permutation in which only two elements are in a different order, that is, for which $\sigma(i) = i$ for all $i = 1, \ldots, n$ except two.

Example 14.11. A permutation can be denoted in several different ways:

$$(1\ 2)(4\ 6\ 5) \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 6 & 4 & 5 \end{pmatrix} \qquad \begin{array}{ccc} 1 & \mapsto & 2 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 3 \\ 4 & \mapsto & 6 \\ 5 & \mapsto & 4 \\ 6 & \mapsto & 5 \end{array}$$

all describe the same permutation. Moreover, $(1\ 2)(4\ 6\ 5)$ is the same as $(1\ 2)(6\ 5)(4\ 6)$, if we apply the 2-element permutations from right to left:

$$\begin{array}{ccccc} & (4\ 5\ 6) & & (1\ 2) & \\ 1 & \mapsto & 1 & \mapsto & 2 \\ 2 & \mapsto & 2 & \mapsto & 1 \\ 3 & \mapsto & 3 & \mapsto & 3 \\ 4 & \mapsto & 6 & \mapsto & 6 \\ 5 & \mapsto & 4 & \mapsto & 4 \\ 6 & \mapsto & 5 & \mapsto & 5 \end{array} \qquad \begin{array}{ccccccc} & (4\ 6) & & (6\ 5) & & (1\ 2) & \\ 1 & \mapsto & 1 & \mapsto & 1 & \mapsto & 2 \\ 2 & \mapsto & 2 & \mapsto & 2 & \mapsto & 1 \\ 3 & \mapsto & 3 & \mapsto & 3 & \mapsto & 3 \\ 4 & \mapsto & 6 & \mapsto & 6 & \mapsto & 6 \\ 5 & \mapsto & 5 & \mapsto & 4 & \mapsto & 4 \\ 6 & \mapsto & 4 & \mapsto & 5 & \mapsto & 5 \end{array}$$

On a set of size n there are $n!$ permutations and $n(n-1)/2$ transpositions. They are related to each other, but in a difficult to prove way.

Theorem 14.12. *Every permutation on a set of n elements may be uniquely (up to rearrangement) described as a composition of transpositions.*

This is a nontrivial fact and we do not prove it here.

Definition 14.13: Let $A \in \mathcal{M}_{n\times n}$, and let σ be a permutation on a set of size n.

- The *parity* of σ is odd or even depeding on if the number of transpositions necessary to represent it is odd or even.
- The *sign* of σ is $+1$ if the parity of σ is even, and -1 is the parity of σ is odd. This number is denoted by $\operatorname{sgn}(\sigma)$.
- The *determinant* of a matrix A can be defined using permutations of the columns of A. That is,

$$\det(A) = \sum_{\text{permutations } \sigma} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)} = \sum_{\text{permutations } \sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)}. \quad (6)$$

Inquiry 14.14: Recall from Definition 3.7 that elementary matrices are either *permutation* (swaps rows), *elimination* (adds multiples of rows), or *diagonal* (multiplies rows by a number) matrices.

- What is the determinant of any elimination matrix?
- What is the determinant of any diagonal matrix?
- What is the determinant of a permutation matrix that swaps two rows? What about three, four rows? Start with some small examples to see what happens.

Convince yourself that permutation matrices with an *odd* number of row swaps have determinant -1, and permutation matrices with an *even* number of row swaps have determinant 1. This is the concept of *parity*.

Example 14.15. There are $3! = 6$ permutations on a set of size 3, so a determinant of a 3×3 matrix is an alternating sum of 6 terms. The permutations are given below.

$$\begin{array}{ccc} & \rho & \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 3 \end{array} \quad \begin{array}{ccc} & \sigma & \\ 1 & \mapsto & 2 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 3 \end{array} \quad \begin{array}{ccc} & \tau & \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 1 \end{array} \quad \begin{array}{ccc} & \mu & \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 2 \end{array} \quad \begin{array}{ccc} & \nu & \\ 1 & \mapsto & 2 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 1 \end{array} \quad \begin{array}{ccc} & \lambda & \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 2 \end{array}$$

The transpositions are σ, τ, μ. Note that $\nu = \tau \circ \sigma$ and $\lambda = \sigma \circ \tau$, which gives us a complete description of the signs of these permutations:

permutation σ	ρ	σ	τ	μ	ν	λ
$\text{sgn}(\sigma)$	1	-1	-1	-1	1	1

So if $A = \begin{bmatrix} 4 & -2 & 1 \\ 7 & 0 & 3 \\ -1 & -3 & 4 \end{bmatrix}$, then the determinant is

$$\begin{aligned}\det(A) &= A_{1\rho(1)}A_{2\rho(2)}A_{3\rho(3)} - A_{1\sigma(1)}A_{2\sigma(2)}A_{3\sigma(3)} + \cdots + A_{1\lambda(1)}A_{2\lambda(2)}A_{3\lambda(3)} \\ &= 4 \cdot 0 \cdot 4 - \cdot(-2) \cdot 7 \cdot 4 + \cdots + 1 \cdot 7 \cdot (-3) \\ &= 77.\end{aligned}$$

However, if we had a different matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 7 & 0 & 3 \\ 0 & -3 & 4 \end{bmatrix}$, then all permutations except one would have a factor of zero in them. That is, since the product $A_{1\sigma(1)}A_{2\sigma(2)} \cdots A_{n\sigma(n)}$ has exactly one element in each row and exactly one element in each column, none of the terms in the combinatorial definition of the determinant can have two elements in the same row or in the same column. In other words,

$$\det(A') = \text{sgn}(\mu) \cdot 4 \cdot 3 \cdot (-3) = (-1) \cdot (-36) = 36.$$

Taking 4 in row 1, column 1, we cannot take any other element in column 1, so we must take row 2, column 3, to get a nonzero number. That leaves row 3, column 2 as the final factor (since columns 1 and 3 have already been used). All other terms in the expansion (6) will have at least one factor of 0, so can be safely ignored.

14.3 Exercises

Exercise 14.1. Show with a counter example that the set of all invertible $n \times n$ matrices is not a subspace of $\mathcal{M}_{n\times n}$. That is, show it is not a vector space.

Exercise 14.2. Recall the definition of an inverse of a matrix A, which was a matrix B such that $AB = BA = I$. Show that the statement $AB = I$ implies $BA = I$.

Exercise 14.3. Let $A \in \mathcal{M}_{n\times n}$. Show that $\det(A) = 0$ is equivalent to saying that there is a nonzero vector $\mathbf{x}$ for which $A\mathbf{x} = 0$.

Exercise 14.4. How many cofactors, or minors, of the matrix below are nonzero? How many terms in the recursive definition of the determinant are nonzero?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Exercise 14.5. Find the parity of the two permutations below.

$$\begin{array}{ccc} & \sigma & \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 2 \\ 4 & \mapsto & 4 \end{array} \qquad \begin{array}{ccc} & \rho & \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 2 \\ 4 & \mapsto & 4 \end{array}$$

Use this to find the determinant of the matrix $A = \begin{bmatrix} 7 & 0 & -1 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Exercise 14.6. 1. Using the permutation formula, compute the determinant of

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

2. How many nonzero entries can a $n \times n$ matrix have so that the permutation formula has only one nonzero term?

Lecture 15: Properties of the determinant

Chapters 5.2,5.3 in Strang's "Linear Algebra"

- Fact 1: The determinant of a product is the product of determinants: $\det(AB) = \det(B)\det(A)$
- Fact 2: The determinant is the product of the pivots, up to a sign change.
- Fact 3: The determinant is nonzero iff the matrix is invertible.

- Skill 1: Compute determinants of products, inverses, transposes of matrices.
- Skill 2: Apply the properties of the determinant.

This lecture explores some properties of the determinant.

15.1 Splitting the determinant

We begin by showing that the determinant is *multiplicative*, that is, that $\det(AB) = \det(A)\det(B)$ for any $n \times n$ matrices A, B. First we need to revisit elementary matrices in Definition 3.7.

Lemma 15.1. Let $A \in \mathcal{M}_{n\times n}$ be an invertible matrix. That is, A^{-1} exists.

- If P is a permutation matrix of a single row swap, then $\det(PA) = \det(P)\det(A) = -\det(A)$.
- If E is an elimination matrix, then $\det(EA) = \det(E)\det(A) = \det(A)$.
- If D is a diagonal matrix, then $\det(DA) = \det(D)\det(A)$.

Proof. The first point follows from the alternating property from Proposition 14.7 and the thrid point of Inquiry 14.14.

The second point follows by multilinearity from Proposition 14.7 and the first point ofInquiry 14.14, which gives that $\det(E) = 1$. Elimination matrices are row operations, so in terms of A and the rows $\mathbf{r}_1, \dots, \mathbf{r}_n$ of A,

$$\begin{aligned}
\det(EA) &= \det(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j - \ell_{ij}\mathbf{r}_i, \dots, \mathbf{r}_n) \\
&= \underbrace{\det(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_n)}_{\det(A)} - \ell_{ij} \underbrace{\det(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_i, \dots, \mathbf{r}_n)}_{0 \text{ because two rows the same}} \\
&= \det(A).
\end{aligned}$$

The third point follows by the second point of Inquiry 14.6, which says that $\det(D)$ is the product of its diagonal entries, and by the recursive definition of the determinant. If D has all ones on the diagonal except on row i, then

$$\begin{aligned}
\det(DA) &= \sum_{j=1}^{n} (-1)^{i+j} (DA)_{ij} \det((DA)^{ij}) \\
&= D_{ii} \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det((DA)^{ij}) \\
&= D_{ii} \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(A^{ij}) \\
&= D_{ii} \det(A) \\
&= \det(D)\det(A).
\end{aligned}$$

If D has more than one diagonal entry that is not 1, repeat this step for every such row. □

Inquiry 15.2: Let $A \in \mathcal{M}_{3\times 3}$.

- Suppose that row operations turn A into $\left[\begin{smallmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. Explain why A has determinant zero.
- Suppose that A has 3 pivots. Explain why A has a nonzero determinant.

The above inquiry sets the scene for the following inquiry and the proposition afterward.

Inquiry 15.3: Let $A, B \in \mathcal{M}_{n\times n}$.

- Suppose that $\det(A) = 0$. Show by contradiction that $\det(AB)$ must also be 0.
- Suppose that $\det(A) \neq 0$ and $\det(B) \neq 0$. Show that $\det(AB) = \det(A)\det(B)$.

Hint: For the second point, convince yourself that A having n pivots means A can be expressed as a product of elementary matrices.

We conclude this section with a strong relationship among some big concepts we have seen so far: pivots, invertibility, and the determinant.

Proposition 15.4. Let $A \in \mathcal{M}_{n\times n}$.

- The determinant of A is the product of the pivots of A, up to a sign change.
- The determinant of A is nonzero if and only if A has n pivots.
- The determinant is zero if and only if A is not invertible.

Proof. The first point follows from the second point of Inquiry 15.3. The second point is a direct consequence of the first point. The third point follow from both points of Inquiry 15.3. □

15.2 Inverses and transposes

Now we take a look at how the determinant works with transposes and inverses.

Proposition 15.5. Let $A \in \mathcal{M}_{n\times n}$ be invertible (that is, have nonzero determinant).

- The determinant of the tranpose is the same as the determinant: $\det(A^T) = \det(A)$
- The determinant of the inverse is the reciprocal of the determinant: $\det(A^{-1}) = \det(A)^{-1}$

Proof. The first statement follows by using the fact that if A is invertible, then it may be expressed as the product of elementary matrices. Using the properties of the transpose (after Definition2.9, the transpose of the product is the (reversed) product of the individual factors. Finally by applying multiplicativity of the determinant, we get back the original matrix A.

The second statement follows from Proposition 15.1 and the fact that $AA^{-1} = I$:

$$\begin{aligned} AA^{-1} = I \quad &\Longrightarrow \quad \det(AA^{-1}) = \det(I) \\ &\Longrightarrow \quad \det(A)\det(A^{-1}) = 1 \\ &\Longrightarrow \quad \det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}. \end{aligned}$$

□

Recall from Definition 14.4 the ij-*minor* of a matrix A was the determinant of the submatrix after the ith row and jth column are removed. The ij-*cofactor* was the ij-minor multiplied by $(-1)^{i+j}$.

Proposition 15.6. Let $A \in \mathcal{M}_{n\times n}$ be invertible, and let $C_{ij} = (-1)^{i+j}\det(A^{ij})$ be the ij-cofactor of A. Then the ij-entry in the inverse is

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det(A)}.$$

In general, for C the cofactor matrix of A, we have $AC^T = \det(A)I$, or $A^{-1} = C^T/\det(A)$.

Proof. This comes from the recursive definition of the determinant, which states that

$$\begin{aligned}\det(A) &= A_{11}C_{11} + A_{12}C_{12} + \cdots + A_{1n}C_{1n} = \mathbf{a}_1^T\mathbf{c}_1,\\ \det(A) &= A_{21}C_{21} + A_{22}C_{22} + \cdots + A_{2n}C_{2n} = \mathbf{a}_2^T\mathbf{c}_2,\end{aligned}$$

and so on, where $\mathbf{a}_i$ is the ith row of a and $\mathbf{c}_i$ is the ith row of C. Moreover, for $i \neq j$, the sum

$$\det(A') = A_{i1}C_{j1} + A_{i2}C_{j2} + \cdots + A_{in}C_{jn} = \mathbf{a}_i^T\mathbf{c}_j$$

of some new matrix A' must be zero, as this is the determinant for a matrix whose ith and jth rows are the same. That is, A_{j1} does not appear in C_{j1}, so having $A_{j1} = A_{i1}$ is allowed for this determinant. Inquiry 14.9 told us that a matrix with two equal rows has determinant zero. Hence

$$\underbrace{\begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ & \vdots & \\ - & \mathbf{a}_n & - \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & & | \end{bmatrix}}_{C^T} = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix},$$

or $AC^T = \det(A)I$. □

This formula generalizes the formula for the inverse of the 2×2 matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $A^{-1} = \frac{1}{ad-bc}\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$. The determinant is still in the denominator, but the cofactors come from larger matrices and so the inverse is not just about rearranging elements.

Example 15.7. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 2 & 8 & 0 \\ 7 & 2 & 9 & 3 & 6 \\ 0 & 0 & 0 & 3 & 1 \end{bmatrix}, \qquad \det(A) = -136.$$

This matrix is invertible, and the $(4,4)$-entry of the inverse will be

$$(A^{-1})_{44} = \frac{(-1)^{4+4}}{-136}\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{-1}{68}.$$

A final application of the determinant that we will see is in a physical setting. Recall the *standard basis* from Example 7.7 in Lecture 7.

Definition 15.8: Let $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \in \mathbf{R}^n$, arranged as columns of $A \in \mathcal{M}_{n\times(n-1)}$, and let $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbf{R}^n$ be the standard basis vectors. The *cross product* of the vectors $\mathbf{v}_i$ is the vector

$$X(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) := \sum_{i=1}^{n} (-1)^{i+n} \det(A^i)\mathbf{e}_i = \begin{vmatrix} | & | & & | & \mathbf{e}_1 \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_{n-1} & \vdots \\ | & | & & | & \mathbf{e}_n \end{vmatrix},$$

where $A^i \in \mathcal{M}_{(n-1)\times(n-1)}$ is A with the ith row removed. The expression on the right is a formal determinant, since we can't put in a whole vector $\mathbf{e}_i$ in a single entry.

Example 15.9. What does the cross product represent? In three dimensions, it is the *right-hand rule* of physicists, determining the direction a moving charge from a rotating magnetic field. The vector computed will be perpendicular to the initial vectors:

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = (-1)^{1+3} \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}.$$

Remark 15.10. The cross product has several interesting properties:

- $X(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) = 0$ iff the set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ is linearly dependent
- For $n = 2$, the length of the cross product is $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\||\sin(\theta)|$
- The cross product is related to the dot product by $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$
- The cross product is *anti-commutative*, or *skew-symmetic*: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

Inquiry 15.11: This inquiry is about the cross product.

- Compute the cross product of the two basis vectors for the plane defined by $z = 10x - 2y$ (see Example 8.2).
- Compare your answer above with a normal vector to this plane. Are the two vectors the same? Are they similar?
- You should have four vectors from the two points above. Explain why their span can be expressed using at most three vectors.

15.3 Exercises

Exercise 15.1. Show that the cross product $X(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$ is *skew-symmetric*, in the sense that swapping the order of two entries puts a negative sign in front.

Lecture 16: Defining eigenvalues and eigenvectors

Chapter 6.1 in Strang's "Linear Algebra"

- Fact 1: An $n \times n$ matrix has at most n eigenvalues, which may be real or complex.
- Fact 2: The roots of the characteristic polynomial $\det(A - \lambda I)$ are the eigenvalues of A.

- Skill 1: Find eigenvectors and eigenvalues of a matrix
- Skill 2: Given only eigenvalues and eigenvectors of A, compute $A\mathbf{x}$ for any $\mathbf{x}$
- Skill 3: Given only eigenvalues and eigenvectors, construct a matrix with these eigenvalues and eigenvectors

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This lecture gets to the heart of the current topic of *eigensystems*. Eigenvalues are important to undrstand what a matrix does to vectors. Eigenvectors are unique in that their direction does not change when multiplied by a matrix A (though their length may change).

16.1 Words beginning with "eigen"

It is important to remember that eigenvalues are unique, but eigenvectors are not, as they can be multiplied by any real number.

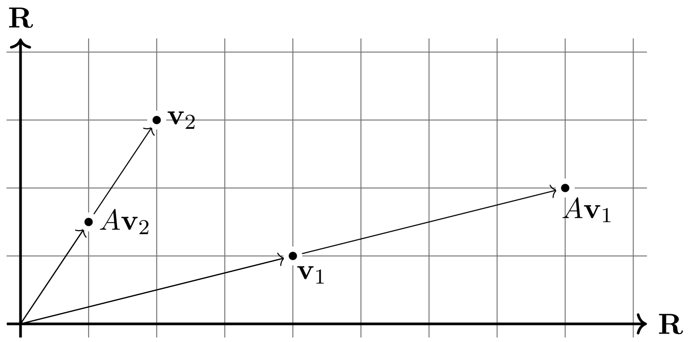
Definition 16.1: Let $A \in \mathcal{M}_{n\times n}$. For every vector $\mathbf{v}$ with $A\mathbf{v} = \lambda\mathbf{v}$, where $\lambda \in \mathbf{R}$,

- the vector $\mathbf{v}$ is called an *eigenvector*,
- the value λ is called the *eigenvalue*,
- the pair $(\mathbf{v}, \lambda)$ is called an *eigenpair*.

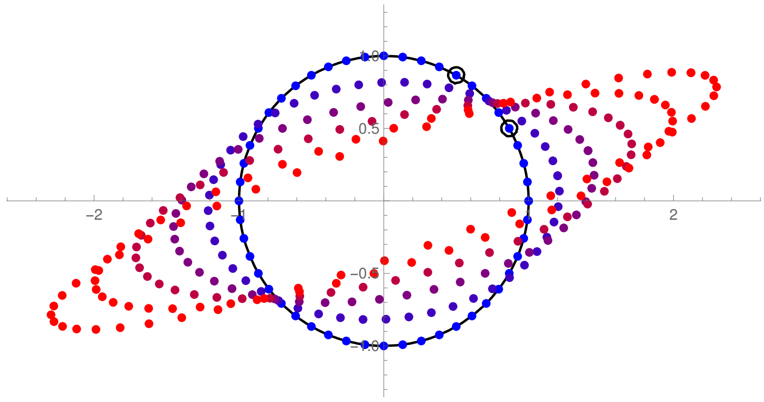
The set of all eigenvalues of A is called the *spectrum* of A. The set of all eigenpairs whose eigenvectors are linearly independent is called the *eigensystem* of A. Eigensystems are unique up to vector scaling.

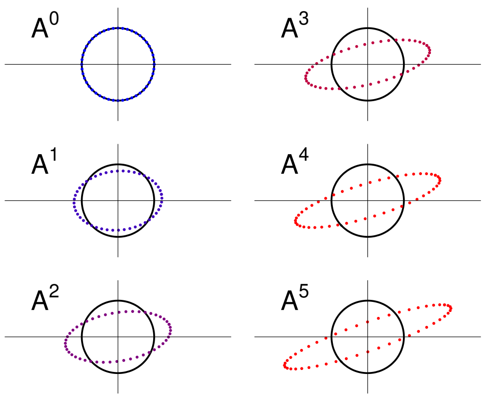
Eigenvectors describe the direction in which a matrix changes $\mathbf{R}^n$, and the eigenvalues describe the stretching that is done in that direction.

Example 16.2. In $\mathbf{R}^2$, the matrix $A = \begin{bmatrix} 23/10 & -6/5 \\ 9/20 & 1/5 \end{bmatrix}$ has eigenvector $\mathbf{v}_1 = \left[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\right]$ with eigenvalue 2, and eigenvector $\mathbf{v}_2 = \left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right]$ with eigenvalue $\frac{1}{2}$.



The vector $\mathbf{v}_1$ gets longer and $\mathbf{v}_2$ gets shorter as A is applied more times. Adjusting $\mathbf{v}_1$ and $\mathbf{v}_2$ so that they make angles $\frac{\pi}{6}$ and $\frac{\pi}{3}$ with the x-axis, respectively, we can visually see what happens to vectors on the unit circle as A is applied more times.





The unit eigenvectors are marked with black circles around them. They are also distringuished from other vectors because their "trajectory" as A is applied is a striaght line. Below in Remark 16.8 we see what happens to vectors that are not exactly an eigenvector.

Example 16.3. Consider the following examples of eigenvectors and eigenvalues.

- The matrix $A = \left[\begin{smallmatrix} 4 & -2 \\ 0 & 2 \end{smallmatrix}\right]$ has eigenvector $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ with eigenvalue 2. But A also has eigenvalue $\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]$ with eigenvalue 2.
- The matrix $B = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ has no (real) eigenvalues. This is the rotation matrix with $\theta = \frac{\pi}{2}$. In the second part of this lecture we will see how to get an eigenvalue from this matrix.
- The identity matrix has every vector as an eigenvector with eigenvalue 1.
- The projection matrix $P = \text{proj}_U$ (from Lecture 10) has every vector in U as an eigenvector with eigenvalue 1, and has every vector of $U^\perp$ as an eigenvector with eigenvalue 0.

Eigenvectors $\mathbf{v}, \mathbf{w}$ of a matrix A are called *independent* eigenvectors if the set $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent.

Inquiry 16.4: Let $\mathbf{v} = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right], \mathbf{w} = \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \in \mathbf{R}^2$ be fixed.

- If $A \in \mathcal{M}_{2\times 2}$ with $A\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \mathbf{v}$ and $A\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \mathbf{w}$, what is the determinant of A?
- Suppose that there is $B \in \mathcal{M}_{2\times 2}$ with $\mathbf{v}, \mathbf{w}$ as eigenvectors. In what cases will the vector $\mathbf{v} + \mathbf{w}$ be an eigenvector for B?
- Must there always exist a 2×2 matrix with $\mathbf{v}$ and $\mathbf{w}$ as eigenvectors?. That is, knowing only $\mathbf{v}$ and $\mathbf{w}$, can you construct a 2×2 matrix with these as eigenvectors?

16.2 The characteristic polynomial

So far we have seen just examples of eigenvalues and eigenvectors, but not yet a procedure for finding them. We describe this procedure now.

Definition 16.5: Let $A \in \mathcal{M}_{n\times n}$. The *characteristic polynomial* of A is

$$\chi(t) = \det(A - tI). \tag{7}$$

The roots λ_i of the characteristic polynomial are the eigenvalues of A. The multiplicity of each root λ_i is its *algebraic multiplicity*.

Once the roots $\lambda_1, \ldots, \lambda_k$ of χ are found, then $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ can be solved in each coordinate to find the corresponding eigenvector $\mathbf{v}_i$.

Example 16.6. Consider the matrix $A = \left[\begin{smallmatrix} 2 & 3 \\ -1 & 6 \end{smallmatrix}\right]$. What are its eigenvalues and corresponding eigenvectors? We must solve $\det(A - \lambda I) = 0$:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \begin{vmatrix} 2-\lambda & 3 \\ -1 & 6-\lambda \end{vmatrix} \\ &= (2-\lambda)(6-\lambda) + 3 \\ &= 12 - 8\lambda + \lambda^2 + 3 \\ &= \lambda^2 - 8\lambda + 15 \\ &= (\lambda - 5)(\lambda - 3). \end{aligned}$$

Hence the eigenvalues are $\lambda = 5$ and $\lambda = 3$. To find the corrseponding eigenvectors, we solve:

$$A\mathbf{v} = 3\mathbf{v} \quad \iff \quad \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 3\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \iff \quad \begin{bmatrix} 2v_1 + 3v_2 \\ -v_1 + 6v_2 \end{bmatrix} = \begin{bmatrix} 3v_1 \\ 3v_2 \end{bmatrix}.$$

This is a linear system of 2 equations, which has solution (by back-substitution) $\mathbf{v} = \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$, though we can choose any value we want for v_2 (and we choose 1 - to avoid such problems, we often take eigenvectors with unit length). Similarly, $\lambda = 3$ has the eigenvector $\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$.

Inquiry 16.7: Consider the vectors $\mathbf{v} = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right], \mathbf{w} = \left[\begin{smallmatrix} -1 \\ 3 \end{smallmatrix}\right] \in \mathbf{R}^2$.

- Construct a 2×2 matrix A that has $\mathbf{v}$ as an eigenvector.
- What is the determinant of A? What does that say about its other eigenvalue?
- Construct a 2×2 matrix B that has $\mathbf{v}$ as an eigenvector with eigenvalue 2 and $\mathbf{w}$ as an eigenvector with eigenvalue 3.
- Compute the determinant and trace of B.

Remark 16.8. If $A \in \mathcal{M}_{n\times n}$ has n eigenvectors, then knowing them and their eigenvalues is enough to know the effect of A on any matrix in $\mathbf{R}^n$. In Example 16.6 we found two eigenvalues and two eigenvectors. Then for any other vector we have

$$A\begin{bmatrix} 2 \\ -2 \end{bmatrix} = A\left(2\begin{bmatrix} 3 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2A\begin{bmatrix} 3 \\ 1 \end{bmatrix} - 4A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\cdot 5\begin{bmatrix} 3 \\ 1 \end{bmatrix} - 4\cdot 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ -2 \end{bmatrix}.$$

Definition 16.9: Let $A \in \mathcal{M}_{n\times n}$. For every eigenvalue λ,

- the number of linearly independent eigenvectors with λ as their eigenvalue is the *geometric multiplicity* of λ,
- the span of these linearly independent eigenvectors is the *eigenspace* of λ.

In other words, if A has k eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in its eigensystem, then the number of $\mathbf{v}_i$ with eigenvalue λ is the geometric multiplicity of λ.

Inquiry 16.10: Consider the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

- Compute the eigensystem of A.
- What are the eiegenspaces of A?
- Explain the relationship between the dimension of an eigenspace and its geometric multiplicity.

16.3 Exercises

Exercise 16.1. Consider the matrix $A = \begin{bmatrix} 6 & -5 \\ 5 & -2 \end{bmatrix}$.

1. Find the eigenvalues and eigenvectors of A. Be careful, there may be complex numbers!
2. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors, compute the dot product $\mathbf{v}_1 \cdot \mathbf{v}_2$. Is it a complex or a real number?

Exercise 16.2. Let $\mathbf{v} \in \mathbf{R}^n$ be a unit vector, and let $A = \mathbf{v}\mathbf{v}^T$.

1. Show that A is a projection matrix.
2. Show that $\mathbf{v}$ is an eigenvector of A and find its eigenvalue.
3. Show that if $\mathbf{u} \perp \mathbf{v}$, then $A\mathbf{u} = 0$.
4. How many independent eigenvectors does A have with eigenvalue 0?

Exercise 16.3. Consider the values $\lambda_1 = -3$, $\lambda_2 = -2$, $\lambda_3 = 5$.

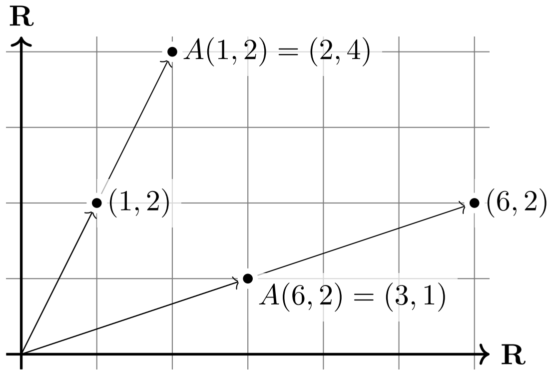
1. Construct two different 3×3 matrices with $\lambda_1, \lambda_2, \lambda_3$ as eigenvalues.
2. What are the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of the two matrices you created in part (a)?
3. If $\lambda_3 = -2$, explain why every linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ is an eigenvector.

Exercise 16.4. You are given that a matrix B has eigenvalues $-1, 2, 5$ and a matrix C has eigenvalues $9, 3, 1$. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 7 \\ 8 & 0 & 3 & 0 & 8 & 0 \\ 0 & 0 & 0 & 9 & -9 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -5 & 2 & 1 \end{bmatrix}.$$

Exercise 16.5. Construct a 2×2 matrix with eigenvector $\begin{bmatrix} x \\ y \end{bmatrix}$ having eigenvalue λ, and eigenvector $\begin{bmatrix} z \\ w \end{bmatrix}$ having eigenvector μ.

Exercise 16.6. Let $A \colon \mathbf{R}^2 \to \mathbf{R}^2$ be the 2×2 matrix for which $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $A \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. This is described in the picture below.



1. What is the eigensystem of A? Express $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ as linear combinations of the eigenvectors of A.
2. Using the task above, compute $A\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Use this to construct the matrix of A.
3. Using eigenvalues, explain why A is invertible.

Lecture 17: Properties of eigenvalues and eigenvectors

Chapter 6.1 in Strang's "Linear Algebra"

- Fact 1: The sum of the eigenvalues is the trace of the matrix.
- Fact 2: The product of the eigenvalues is the determinant of the matrix.

- Skill 1: Do computations with eigenvalues and eigenvectors.
- Skill 2: Compute eigensystems of special matrices.

We continue understading the key properties of eigenvalues and eigenvectors.

17.1 Properties of eigensystems

Recall that the key idea of the eigensystem of a matrix $A \in \mathcal{M}_{n\times n}$ was that it explains how $\mathbf{R}^n$ is transformed, when A multiplies any vector in $\mathbf{R}^n$.

Definition 17.1: Let $A \in \mathcal{M}_{n\times n}$. If there are vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$ for which there exists $\lambda, \mu \in \mathbf{R}$, such

- $A\mathbf{v} = \lambda\mathbf{v}$, then $\mathbf{v}$ is called an *eigenvector*, or *right eigenvector* of A,
- $\mathbf{w}^T A = \mu\mathbf{w}^T$, then $\mathbf{w}$ is called a *left eigenvector* of A.

Note that a right eigenvector of A is a left eigenvector of A^T.

If no adjective "right" or "left" is used, then "right" is assumed. The relationship between left and right eigenpairs is not immediate.

Inquiry 17.2: Let $A \in \mathcal{M}_{n\times n}$.

- Suppose that A is symmetric. If $(\mathbf{v}, \lambda)$ is an eigenpair for A, show that $\mathbf{v}$ is an eigenvector for $A^T A$. What is its eigenvalue?
- Suppose that there are n distinct eigenpairs $(\mathbf{v}_i, \lambda_i)$ for A, with each eigenvector being both a right and a left eigenvector. Show that $AA^T = A^T A$.

Remark 17.3. Let $A \in \mathcal{M}_{n\times n}$ have eigenvalues $\lambda_1, \ldots, \lambda_n$ (not all necessarily distinct). The characteristic polynomial can then be expressed as

$$\chi(t) = (-1)^n (t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n).$$

This follows from the definition of the characteristic polynomial and the recursive definition of the determinant. The coefficient $(-1)^n$ comes from the fact that $-t$ is multiplied by itself n times, and so the leading term must be $(-1)^n t^n$.

Proposition 17.4. Let $A \in \mathcal{M}_{n\times n}$.

- The eigenvalues of A and A^T are the same, but not necessarily their eigenvectors.
- If A is upper or lower triangular, its eigenvalues are on its diagonal.
- If the rank of A is less than n, then A has an eigenvalue 0 for a non-trivial eigenvector.
- If A has an eigenpair $(\mathbf{v}, \lambda)$, then A^n has an eigenpair $(\mathbf{v}, \lambda^n)$.

Proof. The first point follows by distributing transposes in a sum (see Remark 4.14) in

$$\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - (\lambda I)^T) = \det(A^T - \lambda I),$$

so the characteristic polynomial, and hence the eigenvalues, of A and A^T are the same.

The second point follows by using the standard basis of $\mathbf{R}^n$ as eigenvectors.

The third point follows by using a vector in the nullspace.

The fourth point follows from a repeated application of $A\mathbf{v} = \lambda\mathbf{v}$:

$$A^n\mathbf{v} = A^{n-1}(A\mathbf{v}) = A^{n-1}(\lambda\mathbf{v}) = \lambda A^{n-2}(A\mathbf{v}) = \lambda^2 A^{n-3}(A\mathbf{v}) = \cdots = \lambda^n\mathbf{v}.$$

We are allowed to move the λ from the right to the left of A^{n-1} because λ is a number. □

The first point above is similar to the determinant, however: row operations change the eigenvalues (they do not change the determinant). The sum of the diagonal entries in a matrix is called the *trace* of the matrix.

Inquiry 17.5: Recall that the characteristic polynomial of $A \in \mathcal{M}_{n\times n}$ is $\chi(t) = \det(A - tI)$. For this question, let $A = \left[\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{smallmatrix}\right]$.

- The recursive definition of $\det(A - tI)$ has 6 terms in its expansion. Write all of these out, without expanding the $(a_{ij} - t)$ factors.
- When the $(a_{ij} - t)$ factors are all expanded,
  - what is the coefficient of t^3?
  - what is the coefficient of t^2?
  - what is the coefficient of t?
  - what is the constant term?
- Among the parts above, find the trace and the determinant.
- Express the characteristic polynomial using the trace and the determinant.

How do you think this generalizes to higher $n \in \mathbf{N}$?

17.2 Complex numbers

Sometimes we come across matrices (as in Example 16.3) that do not seem to have eigenvalues, such as $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Its characteristic polynomial is $\chi(t) = t^2 + 1$. This polynomial has no real solutions, but does have *complex* solutions.

Definition 17.6: The *complex numbers* $\mathbf{C}$ are elements the set $\mathbf{R} \times \mathbf{R}$, expressed as $a + bi$, $a, b \in \mathbf{R}$, with a new operation:

$$(0,1) \bullet (0,1) = (-1,0) \qquad \Longleftrightarrow \qquad i \cdot i = -1.$$

Remark 17.7. Here are some key properties of the complex numbers .

- multiplying a complex number by i is "rotating the vector by 90 degrees"
- every polynomial with real (or complex) coefficients has roots in the complex numbers

The last statement says that $\mathbf{C}$ is *algebraically closed*.

Inquiry 17.8: Let $A \in \mathcal{M}_{n\times n}$ be a skew-symmetric matrix.

- Compute the eigensystem for $A = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$. How many complex and how many real eigenvalues does A have?
- Compute the eigensystem for $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$. How many complex and how many real eigenvalues does A have? You may use a computer.
- If n is an odd number, explain why A has at least one real root. *Hint: use limits.*
- How many real and how many complex values will a skew-symmetric $n \times n$ matrix have? Begin by showing that $\|A\mathbf{v}\|^2 = -\lambda^2\|\mathbf{v}\|^2$ for any eigenpair $(\mathbf{v}, \lambda)$ of A.

More about complex numbers is discussed in Lecture 24.

Proposition 17.9. Let $A, B \in \mathcal{M}_{n\times n}$.

- The eigenvectors of $A + B$ can not be expressed in terms of the eigenvectors of A and B.
- A and B have the same eigenvectors iff A and B commute (that is, $AB = BA$).

17.3 Exercises

Exercise 17.1. Let $A \in \mathcal{M}_{n\times n}$ and let $\chi(t)$ be its charactristic polynomial.

1. Show that $\chi(0) = (-1)^n \det(A)$. That is, show that the constant term in $\chi(t)$ is $(-1)^n$ times the determinant of A.
2. Show that the coefficient of t^{n-1} in $\chi(t)$ is $-\text{trace}(A)$.

Lecture 18: Diagonalization

Chapter 6.2 in Strang's "Linear Algebra"

- Fact 1: An $n \times n$ matrix has exactly n eigenvalues, counting multiplicity.
- Fact 2: Eigenvalues may be zero. Eigenvectors cannot be the zero vector.
- Fact 3: Although there are n eigenpairs, their eigenvectors may not always form a linearly independent set.

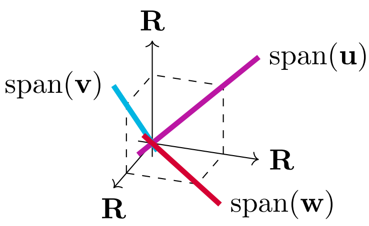
- Skill 1: Diagonalize a matrix with linearly independent eigenvectors.
- Skill 2: Identify matrices that do not have linearly independent eigenvectors.
- Skill 3: Find eigenvalues and eigenvectors of matrices similar to A.

The goal of this section is to reveal within each matrix a *diagonal matrix*. Diagonal matrices are easier to deal with, because they act like numbers rather than matrices. That is, multiplication and all other operations are much easier.

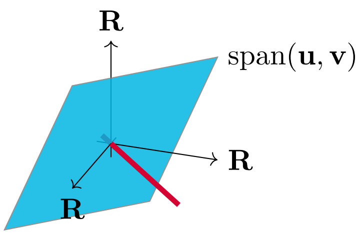
18.1 Multiplicity and diagonalization

We begin with considering several different possibilites of eigenpairs for a 3×3 matrix.

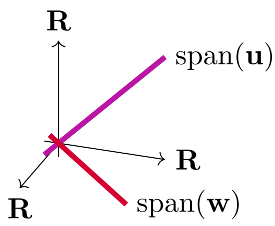
Example 18.1. Consider the three linearly independent vectors $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$



eigenspaces of A



eigenspaces of B



eigenspaces of C

These three vectors may appear in six different ways as eigenvectors of a 3×3 matrix.

- $A = \begin{bmatrix} 1 & 5 & -5 \\ 2 & 4 & -2 \\ -3 & 3 & -1 \end{bmatrix}$ has 3 different eigenvalues, 3 different eigenvectors: $(2, \mathbf{u}), (-4, \mathbf{v}), (6, \mathbf{w})$
- $B = \begin{bmatrix} 1 & 5 & -5 \\ 5 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix}$ has 2 different eigenvalues, 3 different eigenvectors: $(-4, \mathbf{u}), (-4, \mathbf{v}), (6, \mathbf{w})$
- $C = \begin{bmatrix} 1 & 5 & -5 \\ 6 & 0 & -4 \\ 1 & -1 & -3 \end{bmatrix}$ has 2 different eigenvalues, 2 different eigenvectors: $(-4, \mathbf{u}), (6, \mathbf{w})$
- $D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ has 1 eigenvalue, 3 different eigenvectors: $(-4, \mathbf{u}), (-4, \mathbf{v}), (-4, \mathbf{w})$
- $E = \begin{bmatrix} -5 & 1 & 1 \\ 0 & -4 & 0 \\ -1 & 1 & -3 \end{bmatrix}$ has 1 eigenvalue, 2 different eigenvectors: $(-4, \mathbf{v}), (-4, \mathbf{w})$
- $F = \begin{bmatrix} -3 & 1 & -1 \\ 2 & -4 & 0 \\ 3 & 1 & -5 \end{bmatrix}$ has 1 eigenvalue, 1 eigenvector: $(-4, \mathbf{u})$

For B and C, $\lambda = -4$ has algebraic multiplicity 2. For D, E, F, it has algebraic multiplicity 3.

Inquiry 18.2: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^3$ be distinct vectors. Explain why the following situations each cannot happen. Justify your reasoning with the matrix equation $A\mathbf{v} = \lambda\mathbf{v}$, for an eigenpair $(\lambda, \mathbf{v})$.

- A is a 2×3 matrix with eigensystem $\{(1, \mathbf{u}), (2, \mathbf{v})\}$.
- B is a 3×3 matrix with determinant zero and eigensystem $\{(1, \mathbf{u}), (2, \mathbf{v})\}$.
- C is a 3×3 matrix with trace zero and eigensystem $\{(1, \mathbf{u}), (2, \mathbf{v})\}$.
- D is a 3×3 matrix with eigensystem $\{(1, \mathbf{u}), (2, \mathbf{u})\}$.
- E is a 3×3 matrix with eigensystem $\{(1, \mathbf{u}), (2, \mathbf{v}), (3, \mathbf{u} + \mathbf{v})\}$.
- F is a 3×3 matrix with eigensystem $\{(0, \mathbf{u}), (0, \mathbf{v}), (1, \mathbf{w})\}$ and a 2-dimensional column space.

We continue with an example by constructing a matrix from the eigenvectors.

Example 18.3. Let $\mathbf{u} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $\mathbf{v} = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \in \mathbf{R}^2$, which are linearly indepedent vectors. Let $A = \left[\begin{smallmatrix}a & b\\c & d\end{smallmatrix}\right]$ be a matrix with these two as eigenvectors, and corresponding eigenvalues $2, 3$, respectively. What are the entries a, b, c, d of A? We know that

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}, \quad \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}, \quad \Longrightarrow \quad \begin{aligned}a+b &= 2\\ c+d &= 2\\ b &= 0\\ d &= 3\end{aligned} \quad \Longleftrightarrow \quad \begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}2 & 0\\-1 & 3\end{bmatrix}.$$

The equations $A\mathbf{u} = 2\mathbf{u}$ and $A\mathbf{v} = 3\mathbf{v}$ on the left, which can be combined into a single equation

$$A\underbrace{\begin{bmatrix}| & |\\ \mathbf{u} & \mathbf{v}\\ | & |\end{bmatrix}}_{X} = \begin{bmatrix}| & |\\ 2\mathbf{u} & 3\mathbf{v}\\ | & |\end{bmatrix} = \begin{bmatrix}| & |\\ \mathbf{u} & \mathbf{v}\\ | & |\end{bmatrix}\underbrace{\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}}_{\Lambda} \quad \Longrightarrow \quad A = \underbrace{\begin{bmatrix}| & |\\ \mathbf{u} & \mathbf{v}\\ | & |\end{bmatrix}}_{X}\underbrace{\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}}_{\Lambda}\underbrace{\begin{bmatrix}| & |\\ \mathbf{u} & \mathbf{v}\\ | & |\end{bmatrix}^{-1}}_{X^{-1}}$$

The inverse of X can be constructed because $\mathbf{u}, \mathbf{v}$ are linearly independent, so X has rank 2

Definition 18.4: A matrix $A \in \mathcal{M}_{n\times n}$ is *diagonalizable* if it has n linearly independent eigenvectors. If A is diagonalizable, then the *diagonalization* of A is the decomposition of A as the product

$$A = X\Lambda X^{-1}, \tag{8}$$

for Λ a diagonal matrix and $(\Lambda_{ii}, \mathbf{x}_i)$ an eigenpair of A, for every $i = 1, \dots, n$. The vector $\mathbf{x}_i$ is the ith column of X.

Remark 18.5. The matrix X is not unique, as its columns (the eigenvectors of A) may be scaled by any real number. That is, if $A\mathbf{x} = \lambda\mathbf{x}$, then also $A(c\mathbf{x}) = \lambda(c\mathbf{x})$, so $c\mathbf{x}$ is an eigenvector whenever $\mathbf{v}$ is an eigenvector, for any nonzero $c \in \mathbf{R}$. In terms of diagonalization, if $A = X\Lambda X^{-1}$, continuing from Example 18.3, we could have the eigenvectors $5\mathbf{u}$ and $-7\mathbf{v}$ instead of just $\mathbf{u}$ and $\mathbf{v}$. In that case,

$$X = \begin{bmatrix}| & |\\ 5\mathbf{u} & -7\mathbf{v}\\ | & |\end{bmatrix} = \begin{bmatrix}| & |\\ \mathbf{u} & \mathbf{v}\\ | & |\end{bmatrix}\begin{bmatrix}5 & 0\\0 & -7\end{bmatrix} \quad \Longrightarrow \quad X^{-1} = \left(\begin{bmatrix}| & |\\ \mathbf{u} & \mathbf{v}\\ | & |\end{bmatrix}\begin{bmatrix}5 & 0\\0 & -7\end{bmatrix}\right)^{-1} = \begin{bmatrix}1/5 & 0\\0 & -1/7\end{bmatrix}\begin{bmatrix}| & |\\ \mathbf{u} & \mathbf{v}\\ | & |\end{bmatrix}^{-1},$$

and the decomposition in that case is

$$A = \underbrace{\begin{bmatrix} | & | \\ \mathbf{u} & \mathbf{v} \\ | & | \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -7 \end{bmatrix}}_{X} \Lambda \underbrace{\begin{bmatrix} 1/5 & 0 \\ 0 & -1/7 \end{bmatrix} \begin{bmatrix} | & | \\ \mathbf{u} & \mathbf{v} \\ | & | \end{bmatrix}^{-1}}_{X^{-1}}$$

$$= \begin{bmatrix} | & | \\ \mathbf{u} & \mathbf{v} \\ | & | \end{bmatrix} \Lambda \begin{bmatrix} 5 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ 0 & -1/7 \end{bmatrix} \begin{bmatrix} | & | \\ \mathbf{u} & \mathbf{v} \\ | & | \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} | & | \\ \mathbf{u} & \mathbf{v} \\ | & | \end{bmatrix} \Lambda \begin{bmatrix} | & | \\ \mathbf{u} & \mathbf{v} \\ | & | \end{bmatrix}^{-1},$$

which is the same decomposition as we had previously, with just $\mathbf{u}$ and $\mathbf{v}$. We used the fact that diagonal matrices commute with each other.

Example 18.6. Consider diagonalization for different types of matrices:

- If $A = I_n$, then we the eigenvectors are the standard basis vectors of $\mathbf{R}^n$, and the only eigenvalue is 1. This eigenvalue has *algebraic multiplicity* n, because there are n linearly independent eigenvectors with the same eigenvalue. That is, $A = X = \Lambda = I$.
- If A has all nonzero eigenvalues that are all the same, then A must be a multiple of the identity matrix. Indeed:

$$\Lambda = kI \quad \Longrightarrow \quad A = X^{-1}(kI)X = kX^{-1}IX = kX^{-1}X = kI.$$

- If $A \in \mathcal{M}_{4\times 4}$ has two nonzero eigenvalues and two zero eigenvalues, then A may be diagonalizable, but not always. For example:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix}, \qquad \det(A - \lambda I) = \lambda^2((1-\lambda)(-1-\lambda) - 3) = \lambda^2(-4 + \lambda^2),$$

and the roots of the characteristic polynomial are $\lambda = 0$ and $\lambda = \pm 2$. By solving the appropriate matrix equation, we find the nonzero eigenvector / eigenvalue pairs to be

$$2 \text{ for } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad -2 \text{ for } \begin{bmatrix} 0 \\ 0 \\ -1 \\ 3 \end{bmatrix}.$$

For the zero eigenvalues, the corresponding eigenvector $[\, x\ y\ z\ w \,]^T$ will have $z = 0$ and $w = 0$, but there will be no conditions on x, y, so by convention we choose $\mathbf{e}_1$ and $\mathbf{e}_2$ of the standard basis of $\mathbf{R}^4$ to be the eigenvectors. Diagonalization still works:

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{X^{-1}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix} = A$$

However, this works because we essentially have a diagonal block matrix $\left[\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}\right]$, and the 2×2 matrix B had linearly independent eigenvectors. If we do not have a block matrix form with zero eigenvalues, then we cannot diagonalize. Consider the matrix

$$C = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \det(C - \lambda I) = (1-\lambda)(-1-\lambda) + 1 = \lambda^2,$$

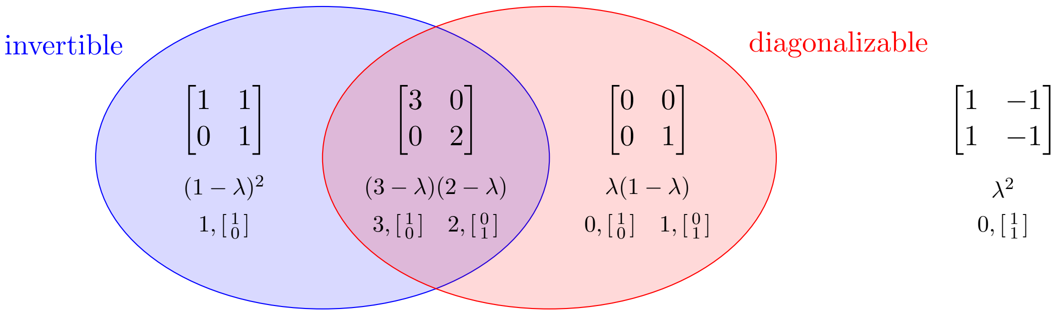
and the roots of the characteristic polynomial are only $\lambda = 0$. The matrix equation to solve is

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \begin{bmatrix} x \\ y \end{bmatrix} \quad \Longleftrightarrow \quad \begin{array}{rcl} x - y & = & 0, \\ x - y & = & 0. \end{array}$$

It seems like the only eigenvector is $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, but then $X = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ does not have full rank and can not be diagonalized.

18.2 Invertibility and similarity

You may be tempted to think that a matrix being *invertible* is the same as being *diagonalizable*, but this is not true. In fact, there is no direct relationship between being invertible and diagonalizable, as the Venn diagram of such matrices below shows.



For eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ of A, invertibility asks whether or not $\lambda_i = 0$. Diagonalizability asks whether or not the $\mathbf{v}_i$ are independent.

> **Inquiry 18.7:** Let $A \in \mathcal{M}_{3\times3}$, and suppose that A has 3 different eigenvalues.
>
> - Explain why A must have 3 linearly independent eigenvectors.
>   *Hint: Show this by contradiciton, assuming that two eigenvectors are linearly independent, and the third is a linear combination of the first two.*
> - If none of the eigenvalues are zero, explain why A is invertible. What happens if one of the eigenvalues is zero?
>   *Hint: Use the diagonalization equation.*
> - Convince yourself that the statement generalizes to any $n \in \mathbf{N}$.

Remark 18.8. Let $A \in \mathcal{M}_{n\times n}$ be diagonalizable, with eigenvector matrix X and corresponding eigenvalue matrix Λ. Then:

- For any invertible $B \in \mathcal{M}_{n\times n}$, the matrix $C = BAB^{-1}$ has the same eigenvalues as A, and has eigenvector matrix BX. Here C and A are called *similar matrices*.
- For any $k \in \mathbf{N}$, the matrix A^k is diagonalizable with the same eigenvectors as A, and with eigenvalues on the diagonal of Λ^k.
- If $|\lambda_i| = |\Lambda_{ii}| < 1$ for all i, then $\lim\limits_{k\to\infty} A^k\mathbf{x} = 0$ for any $\mathbf{x} \in \mathbf{R}^n$.

All of these facts follow directly from the diagonalizing equation $A = X\Lambda X^{-1}$. In the last point, for complex eigenvalues $\lambda = a + bi$, the absolute value is the product of λ with its *conjugate* $\lambda^* = a - bi$:

$$|\lambda| = |a + bi| = (a + bi)(a - bi) = a^2 - (bi)^2 = a - b^2 i^2 = a^2 + b^2.$$

Example 18.9. Consider the matrix $A = \left[\begin{smallmatrix} 1/6 & 1/3 \\ -1/6 & 2/3 \end{smallmatrix}\right]$. The roots of its characteristic polynomial are given by

$$0 = \det(A - \lambda I) = \left(\frac{1}{6} - \lambda\right)\left(-\frac{2}{3} - \lambda\right) + \frac{1}{3} \cdot \frac{1}{6} = \lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} \quad \iff \quad 0 = 6\lambda^2 - 5\lambda + 1,$$

which factors as $0 = (3\lambda - 1)(2\lambda - 1)$, so the eigenvalues are $\lambda_1 = \frac{1}{3}$ and $\lambda_2 = \frac{1}{2}$. By solving the appropriate matrix equations, we get the correspoinding eigenvectors to be $\mathbf{v}_1 = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$ and $\mathbf{v}_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, so the diagonalization of A is

$$A = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}}_{X^{-1}}.$$

The eigenvalues of A^k then are computed by the equation

$$A^k = \left(X\Lambda X^{-1}\right)^k = (X\Lambda X^{-1})(X\Lambda X^{-1}) \cdots (X\Lambda X^{-1}) = X\Lambda(X^{-1}X) \cdots (X^{-1}X)\Lambda X^{-1} = X\Lambda^k X^{-1},$$

and $\Lambda^k = \left[\begin{smallmatrix} 1/3^k & 0 \\ 0 & 1/2^k \end{smallmatrix}\right]$. Hence the eigenvectors of A^k are the same as those for A, and the eigenvalues are siomply powers of the original eigenvalues. We can even construct the matrix A^k explicitly:

$$\begin{aligned} A^k &= X\Lambda^k X^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/3^k & 0 \\ 0 & 1/2^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2/3^k & 1/2^k \\ 1/3^k & 1/2^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{6^k} \begin{bmatrix} 2^{k+1} - 3^k & 2(3^k - 2^k) \\ 2^k - 3^k & 2 \cdot 3^k - 2^k \end{bmatrix} \end{aligned}$$

For example, when $k = 5$, we have

$$A^5 = \frac{1}{7776} \begin{bmatrix} -179 & 422 \\ -211 & 454 \end{bmatrix}.$$

18.3 Exercises

Exercise 18.1. Decompose both matrices below in their $X\Lambda X^{-1}$-decomposition, where Λ is a diagonal matrix with the eigenvalues, and X is the matrix with columns as eigenvectors.

$$A = \begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Exercise 18.2. Let $A \in \mathcal{M}_{3\times 3}$ with the eigenvectors $\left[\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ -1 \\ 0 \end{smallmatrix}\right]$ and eigenvalues $-1, 2, -3$, respectively.

1. Construct the eigenvector matrix X and the eigenvalues matrix Λ.
2. Construct A by the diagonalization equation $A = X\Lambda X^{-1}$.

Exercise 18.3. Diagonalize the matrices A, B below and find what A^k and B^k look like, for any

$k \in \mathbf{N}$. Your answers should have the value k in them.

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 1 \\ 0 & 10 \end{bmatrix}.$$

Exercise 18.4. Let A, B, C be any 3×3 matrices, with C diagonalizable.

1. Show that $\text{trace}(AB) = \text{trace}(BA)$.
2. Use that above to show that $\text{trace}(C)$ is the sum of the three eigenvalues of C.
   Hint: Split up the diagonalization of C into two matrices.
3. Suppose that the eigenvalues of C are $1, \frac{1}{2}, \frac{1}{3}$. Show why the limit $\lim_{n\to\infty} C^n$ exists, and why it has rank 1.

Lecture 19: Special matrices

Chapters 6.4, 6.5 in Strang's "Linear Algebra"

- Fact 1: Symmetric matrices can be decomposed with an orthonormal matrix of eigenvectors.
- Fact 2: Positive definiteness can be expressed in terms of pivots, eigenvalues, determinants, and matix or vector multiplications.

- Skill 1: Apply the results of the spectral theorem
- Skill 2: Express a symmetric matrix as a sum of rank one matrices
- Skill 3: Check if a matrix is positive definite using equivalent properties

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This section is about *symmetric* and *positive definite* matrices. We will see that for any matrix $A \in \mathcal{M}_{m\times n}$, the matrices $A^T A \in \mathcal{M}_{n\times n}$ and $AA^T \in \mathcal{M}_{m\times m}$ are both symmetric and positive definite.

19.1 Symmetric matrices

Recall from Definition 4.2 in Lecture 3 that a matrix $A \in \mathcal{M}_{n\times n}$ is *symmetric* if $A_{ij} = A_{ji}$ for all $1 \leqslant i, j \leqslant n$. This property makes many of the previous computations we did before much easier.

Proposition 19.1 (The Spectral Theorem)**.** Let $A \in \mathcal{M}_{n\times n}$. If A is symmetric, then A has n real eigenvalues and n orthogonal eigenvectors.

This implies that a symmetric matrix can always be diagonalized. Symmetric matrices will often be written "S".

Inquiry 19.2: Consider the symmetric matrix $S = \left[\begin{smallmatrix}1&2&0\\2&1&2\\0&2&1\end{smallmatrix}\right]$.

- Compute the matrices X, Λ for the diagonalization of S.
- Find the matrix B for which BX has orthonormal columns.
- Consider the matrix X' which is the same as X, but with the first two columns swapped. Explain why $X'\Lambda'(X')^{-1}$ is still equal to S. As with X, here Λ' is the same as Λ, but with the first two columns swapped.
- Does column swapping as in the point above work for any symmetric matrix, or only for this particular S?

Example 19.3. Consider $S = \left[\begin{smallmatrix}1&2\\2&4\end{smallmatrix}\right]$. We can find its eigenvalues by solving

$$0 = \det(S - \lambda I) = (1-\lambda)(4-\lambda) = 4 = -5\lambda + \lambda^2 = \lambda(\lambda - 5),$$

for which $\lambda_1 = 0$ and $\lambda_2 = 5$. We find the eigenvectors by solving

$$\begin{bmatrix}1&2\\2&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0\begin{bmatrix}0\\y\end{bmatrix} \iff \begin{array}{rcl}x+2y&=&0\\2x+4y&=&0\end{array} \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\-\frac{1}{2}\end{bmatrix} = \mathbf{v}_1,$$
$$\begin{bmatrix}1&2\\2&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 5\begin{bmatrix}0\\y\end{bmatrix} \iff \begin{array}{rcl}x+2y&=&5x\\2x+4y&=&5y\end{array} \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\1\end{bmatrix} = \mathbf{v}_2.$$

These vectors are orthogonal as $\mathbf{v}_1 \bullet \mathbf{v}_2 = 0$. They both have length $\sqrt{5}/2$, so the normalized vectors are

$$\mathbf{q}_1 = \begin{bmatrix}2/\sqrt{5}\\-1/\sqrt{5}\end{bmatrix}, \qquad \mathbf{q}_2 = \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}.$$

This gives us the diagonalization as

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}_{Q^T}.$$

Remark 19.4. The fact that $S = Q\Lambda Q^T$, where Q has orthonormal columns, allows us to write S in another way. If $S \in \mathcal{M}_{3\times 3}$, then

$$\begin{aligned} S &= \underbrace{\begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{v} & \mathbf{w} \\ | & | & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} - & \mathbf{u}^T & - \\ - & \mathbf{v}^T & - \\ - & \mathbf{w}^T & - \end{bmatrix}}_{Q^T} \\ &= \underbrace{\begin{bmatrix} | & | & | \\ \lambda_1\mathbf{u} & \lambda_2\mathbf{v} & \lambda_3\mathbf{w} \\ | & | & | \end{bmatrix}}_{Q\Lambda} \begin{bmatrix} - & \mathbf{u}^T & - \\ - & \mathbf{v}^T & - \\ - & \mathbf{w}^T & - \end{bmatrix} \\ &= \lambda_1\mathbf{u}\mathbf{u}^T + \lambda_2\mathbf{v}\mathbf{v}^T + \lambda_3\mathbf{w}\mathbf{w}^T, \end{aligned}$$

which is a sum of 3×3 rank one matrices. This description will be important for Lecture 21.

We finish off the first part of this lecture with another comment about the relationship between pivots and eigenvalues.

Remark 19.5. Let $A \in \mathcal{M}_{n\times n}$. Below are the main facts about pivots and eigenvalues summarized, along with a new one:

- $\det(A) = (\text{product of pivots}) = (\text{product of eigenvalues})$
- $\text{trace}(A) = (\text{sum of eigenvalues})$
- $(\text{number of pivots} > 0) = (\text{number of eigvals} > 0)$ whenever A is symmetric

This last fact is counting multiplicity. It follows from the LDU-decomposition of a symmetric matrix, which turns into LDL^T.

Inquiry 19.6: Consider the symmetric matrix $S = \left[\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right]$. For this inquiry, you may use a computer.

- Compute the eigensystem of S.
- As given, S is not invertible. For what values i, j will changing $S_{ij} = S_{ji}$ to something other than 1, make it invertible?
- The eiegnsystem of S contains only integers. Choose some i, j, and change $S_{ij} = S_{ji}$ to something other than 1 so that the eigensystem again only contains integers.

19.2 Positive definite matrices

The second part of this lecture focuses on special types of symmetric matrices.

Definition 19.7: Let $S \in \mathcal{M}_{n\times n}$ be symmetric. The matrix S is *positive definite* if, equivalently,

- all eigenvalues of S are positive
- $\mathbf{v}^T S\mathbf{v} > 0$ for any nonzero $\mathbf{v} \in \mathbf{R}$.

Weakening the conditions to $\lambda \geqslant 0$ and $\mathbf{v}^T S\mathbf{v} \geqslant 0$ means S is (*positive*) *semidefinite*.

Finding eigenvalues is computationally intensive for large matrices, so we use the relationship with pivots from Remark 19.5 to determine when eigenvalues are positive. This gives several quick ways to determine when a matrix is positive definite.

Example 19.8. The 2×2 symmetric matrix $\left[\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right]$ has pivots $a, c-\frac{b^2}{a}$, so the pivots are positive iff $a>0$ and $ac-b^2>0$. For example, all the symmetric matrices

$$\begin{bmatrix} 1 & 10 \\ 10 & 200 \end{bmatrix}, \qquad \begin{bmatrix} 22 & -3 \\ -3 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

are positive definite because they have positive eigenvalues.

Remark 19.9. To see why the two definitions from Definition 19.7 are equivalent, consider an $n\times n$ positive definite matrix S with eigenvector $\mathbf{v}$ and positive eigenvalue λ. Then

$$S\mathbf{v} = \lambda\mathbf{v} \quad \Longrightarrow \quad \mathbf{v}^T S\mathbf{v} = \lambda \mathbf{v}^T\mathbf{v} = \lambda(v_1^2 + \cdots v_n^2) > 0.$$

Conversely, any $\mathbf{x} \in \mathbf{R}^n$ can be expressed as a linear combination $a_1\mathbf{v}_1 + \cdots a_n\mathbf{v}_n$ of the orthonormal eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ of S. Then by orthonormality of the eigenvectors,

$$\mathbf{x}^T S\mathbf{x} = (a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n)^T(a_1\lambda_1\mathbf{v}_1 + \cdots + a_n\lambda_n\mathbf{v}_n) = a_1^2\lambda_1\|\mathbf{v}_1\|^2 + \cdots + a_n^2\lambda_n\|\mathbf{v}_n\|^2 > 0.$$

Proposition 19.10. The previous remark has some nice consequences:

- If $S, T \in \mathcal{M}_{n\times n}$ are positive definite, then $S+T$ is positive definite.
- If $A \in \mathcal{M}_{m\times n}$ has independent columns, then A^TA is positive definite.

Proof. The first point follows from distributing

$$\mathbf{x}^T(S+T)\mathbf{x} = \mathbf{x}^T S\mathbf{x} + \mathbf{x}^T T\mathbf{x}.$$

The second point comes from rewriting

$$\mathbf{x}^T(A^TA)\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = \|A\mathbf{x}\|^2 > 0.$$

□

The proof of the second claim implies that A^TA (and also AA^T) is always positive semidefinite.

Inquiry 19.11: This inquiry uses Definition 13.1 of an *inner product* from Lecture 13.

- Let $S \in \mathcal{M}_{n\times n}$ be positive definite. Show that $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T S\mathbf{v}$ satisfies all the properties of an inner product on $\mathbf{R}^n$.
- Let $A \in \mathcal{M}_{n\times n}$ be a matrix and $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T A\mathbf{v}$ an inner product. Show that A must be a positive definite matrix.
  Hint: To show A must be symmetric, use the symmetric property of the inner product with the standard basis vectors. To show A must be positive definite, use the positive definite property of the inner product.

Proposition 19.12. Let $S \in \mathcal{M}_{n\times n}$ be symmetric. Then, equivalently,

- S is positive definite
- S has all positive pivots
- S has all positive eigenvalues
- Every top-left submatrix of S has positive determinant
- $\mathbf{x}^T S\mathbf{x} > 0$ for any nonzero $\mathbf{x} \in \mathbf{R}^n$
- There exists $A \in \mathcal{M}_{m\times n}$ with independent columns and $S = A^T A$

Example 19.13. Let's check all the claims above on a simple matrix $S = \left[\begin{smallmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{smallmatrix}\right]$. For the pivots, we quickly row reduce:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

The pivots are $2, 3/2, 4/3$, which are all positive. The eigenvalues are the roots of

$$\det(S - \lambda I) =$$

19.3 Exercises

Exercise 19.1. Let $a \in \mathbf{R}$ be nonzero.

1. Find the eigenvalues of $\left[\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right]$.
2. Find the eigenvalues of $\left[\begin{smallmatrix} 0 & 0 & a \\ 0 & ia & 0 \\ -a & 0 & 0 \end{smallmatrix}\right]$.
3. Using a, construct a 4×4 skew-symmetric matrix that has all imaginary eigenvalues.
4. Construct a 3×3 symmetric matrix that has three pivots a and no zero entries.

Exercise 19.2. Let $A \in \mathcal{M}_{m\times n}$. Show that AA^T and $A^T A$ are both symmetric matrices.

Exercise 19.3. The numbers a, b, c are chosen randomly from the set of integers $\{-3, -2, \dots, 2, 3\}$, with replacement, to create a matrix $A = \left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right]$.

1. What is the probability that A is symmetric?
2. What is the probability that A is positive definite?

Exercise 19.4. Consider the two symmetric matrices below, for $a, b \in \mathbf{R}$:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} b & 2 & 0 \\ 2 & b & 3 \\ 0 & 3 & b \end{bmatrix}.$$

1. Find the pivots for both matrices. For what values of a, b will the pivots be positive?
2. Find the eigenvalues for both matrices. For what values of a, b will the eigenvalues be positive?
3. Find the upper left determinants for both matrices. For what values of a, b will the determinants be positive?
4. Choose some b so that pivots, eigenvalues, determinants are positive. Find the $Q\Lambda Q^T$-decomposition for B.

Lecture 20: Generalizing diagonalizability: Jordan form

Chapter 8.3 in Strang's "Linear Algebra"

- Fact 1: Every square matrix is similar to a square matrix in Jordan normal form.
- Fact 2: Jordan normal gives generalized eigenvectors, which always exist, irrespective of geometric multiplicity.

- Skill 1: Construct the Jordan normal form of a square matrix.
- Skill 2: Find the higher rank generalized eigenvectors when algebraic multiplicity exceeds geometric multiplicity.

In this section we continue with the idea of associating a diagonal matrix to every matrix, but this time the matrix will be *almost* diagonal. This allows us to decompose a matrix when the number of linearly independent eigenvectors is less than the rank.

20.1 Missing eigenvectors

Recall that the characteristic polynomial $\chi(t)$ of $A \in \mathcal{M}_{m\times n}$ has n roots $\lambda_1, \dots, \lambda_n$, some of which may repeat, which are the eigenvalues of A. For each $i = 1, \dots, n$,

- the exponent m of the factor $(\lambda - \lambda_i)^m$ of χ is the *algebraic multiplicity* of λ_i
- the number of linearly independent eigenvectors of A having λ_i as an eigenvalue is the *geometric multiplicity* of λ_i.

We already saw the general relationship among these numbers to be

$$1 \leqslant (\text{geometric multiplicity of } \lambda) \leqslant (\text{algebraic multiplicity of } \lambda) \leqslant \operatorname{rank}(A).$$

Inquiry 20.1: Let $A \in \mathcal{M}_{n\times n}$, and suppose that there is a nonzero number $k \in \mathbf{R}$ with $\det(A - kI) = 0$, and a nonzero vector $\mathbf{x} \in \mathbf{R}^n$ with $A\mathbf{x} = 0$.

- The equation $\det(A - kI) = 0$ means k is an eigenvalue of A. How do you know there has to be an eigenvector assicated to this eigenvalue?
  Hint: If the determinant of a matrix is zero, what does that say about the linear independence of the rows / columns?
- Use the equation $A\mathbf{x} = 0$ to find two different $n \times n$ matrices with $\mathbf{x}$ as an eigenvector.
  Hint: Consider the (somewhat silly) equation $A = A + \lambda I - \lambda I$.

Example 20.2. The matrix $B = \left[\begin{smallmatrix}2&2\\0&2\end{smallmatrix}\right]$ has one eigenvalue $\lambda = 2$ with algebraic multiplicity 2 and geometric multiplicity 1, as

$$\begin{bmatrix}2&2\\0&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\lambda x\\ \lambda y\end{bmatrix} \qquad \Longleftrightarrow \qquad \begin{aligned}2x+2y &= \lambda x\\ 2y &= \lambda y,\end{aligned}$$

meaning $y = 0$ and $\lambda = 2$. The single eigenpair is then $(2, \mathbf{v} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$, or in other words

$$A\mathbf{v} = 2\mathbf{v} \quad \Longleftrightarrow \quad A\mathbf{v} - 2\mathbf{v} = 0 \quad (A - 2I)\mathbf{v} = 0.$$

The eigenspace of $\lambda = 2$ is 1-dimensional, and to fill all of $\mathbf{R}^2$, we need a vector like $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Note that

$$(A-2I)\begin{bmatrix}0\\1/2\end{bmatrix} = \begin{bmatrix}0&2\\0&0\end{bmatrix}\begin{bmatrix}0\\1/2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = \mathbf{v} \quad \Longrightarrow \quad (A-2I)^2\begin{bmatrix}0\\1/2\end{bmatrix} = 0.$$

So the vector $\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$ is *almost* like an eigenvector, but not quite! This motivates the next definition.

Definition 20.3: Let $A \in \mathcal{M}_{n\times n}$ with eigenvalue λ, having

- algebraic multiplicity of λ equal to $a \in \{1,\dots,n\}$, and
- geometric multiplicity of λ equal to $b \in \{1,\dots,b\}$.

A vector $\mathbf{v} \in \mathbf{R}^n$ is a *generalized eigenvector of rank k* associated to λ if

$$(A-\lambda I)^k\mathbf{v} = 0 \qquad \text{and} \qquad (A-\lambda I)^{k-1}\mathbf{v} \neq 0,$$

for some $k \in \{1,\dots,n-b+1\}$. The eigenvectors we have seen so far are all generalized eigenvectors of rank 1. We use the convention that $A^0 = I$ for any nonzero A.

Remark 20.4. There is a relationship among generalized eigenvectors $\mathbf{v}_1,\dots,\mathbf{v}_m$ of $A \in \mathcal{M}_{n\times n}$ associated to a particular eigenvalue λ. If $\mathbf{v}_i$ is a generalized eigenvector of rank i, then there is a *cycle* of generalized eigenvectors given by

$$(A-\lambda I)\mathbf{v}_i = \mathbf{v}_{i-1} \qquad \text{or} \qquad (A-\lambda I)^{m-i}\mathbf{v}_m = \mathbf{v}_i \tag{9}$$

for all $i = 1,\dots,m$. The relationship between the number m and other numbers of a matrix is explored in Inquiry 20.7.

Example 20.5. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{with} \qquad \det(A-tI) = (2-t)^3(1-t).$$

The eigensystem of this matrix is given below.

eigenvalue	eigenvector	alg. mult.	geom. mult.
2	$(1,0,0,0)$	3	1
1	$(0,0,0,1)$	1	1

How do we find the generalized eigenvectors? Although in this case it may be quick to see that we are missing $(0,1,0,0)$ and $(0,0,1,0)$, this is not the case in general. We already know the two (generalized) eigenvectors (of rank 1), so by Equation 9, the generalized eigenvector of rank 2 associated to $\lambda = 2$, is a vector $\mathbf{v}$ for which

$$(A-2I)^2\mathbf{v} = 0 \ \text{ and } \ (A-2I)\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \iff \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We quickly see the solution to be $(0,1,0,0)$, as expected. Similarly, the generalized eigenvector of rank 3 associated to $\lambda = 2$ is a vector $\mathbf{w}$ for which

$$(A-2I)^3\mathbf{w} = 0 \ \text{ and } \ (A-2I)^2\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \ \text{ and } \ (A-2I)\mathbf{w} = \mathbf{v}.$$

Solving the last equation we find that $\mathbf{w} = (0,0,1,0)$, again as expected.

Definition 20.6: Let $A \in \mathcal{M}_{n\times n}$. The *Jordan (normal) form* of A is the matrix

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} \qquad \text{with} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix},$$

for every $i = 1, \ldots, s$. Every $J_1, \ldots, J_s$ is a *Jordan block*, and every $\lambda_1, \ldots, \lambda_n$ is an eigenvalue of A. For every i, the number of Jordan blocks with eigenvalue λ is the geometric multiplicity of λ. Jordan normal form is also known under the names *normal form* or *Jordan canonical form*.

To get the size of each Jordan block, we need to do some more work.

Inquiry 20.7: Let $A \in \mathcal{M}_{10\times 10}$, and suppose it has the eigensystem as given in the table below.

eigenvalue	eigenvector	algebraic multiplicity
-1	$\mathbf{u}$	1
-1	$\mathbf{v}$	1
-1	$\mathbf{w}$	3
3	$\mathbf{x}$	2
3	$\mathbf{y}$	
5/11	$\mathbf{z}$	2

- Fill in the missing entry in the table. What is $\chi(t)$?
- How many Jordan blocks will the Jordan form of A have? What will be their sizes?
- Write the Jordan normal form of A.
- How are the sizes of Jordan blocks of the same eigenvalue related to the eigenvalue's algebraic multiplicity?

In general, given a single eigenpair of $A \in \mathcal{M}_{n\times n}$ and the algebraic multiplicity, how can you know the size of the associated Jordan block? Is it even possible?

Example 20.8. If we have the eigenvector with the highest rank in the cycle, we can generate the others. Consider

$$J = \begin{bmatrix} 2 & 1 & & & & & \\ & 2 & & & & & \\ & & 2 & 1 & & & \\ & & & 2 & 1 & & \\ & & & & 2 & 1 & \\ & & & & & 2 & 1 \\ & & & & & & 2 \end{bmatrix},$$

which has only one eigenvalue $\lambda = 2$, with algebraic multiplicity 7, geometric multiplicity 2 and two Jordan blocks associated to it. The rank 1 eigenvectors are

$$\mathbf{u}_1 = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0 \end{bmatrix}^T, \qquad \mathbf{v}_1 = \begin{bmatrix} 0\,0\,1\,0\,0\,0\,0 \end{bmatrix}^T.$$

To get the higher rank generalized eigenvectors, we check the positions of the 1's above the diagonal. It is immediate that

$$\mathbf{u}_2 = \begin{bmatrix} 0\,1\,0\,0\,0\,0\,0 \end{bmatrix}^T, \qquad \mathbf{v}_2 = \begin{bmatrix} 0\,0\,0\,1\,0\,0\,0 \end{bmatrix}^T, \quad \cdots \quad \mathbf{v}_5 = \begin{bmatrix} 0\,0\,0\,0\,0\,0\,1 \end{bmatrix}^T.$$

The relationship is also $(J - 2I)\mathbf{v}_5 = \mathbf{v}_4$, $(J - 2I)^2\mathbf{v}_5 = \mathbf{v}_3$, and so on.

20.2 Jordan's theorem

Recall that a matrix B is similar to a matrix A if there exists an invertible matrix C with $B = C^{-1}AC$. Here we revisit the idea, making precise the relationship between the matrices A and J.

Remark 20.9. Similar matrices do not have the same eigenvectors, but they do have the same eigenvalues. The eigenvectors of similar matrices are related: If $B = C^{-1}AC$ has eigenvector $\mathbf{x}$ with eigenvalue λ, then

$$B\mathbf{x} = \lambda\mathbf{x} \quad \Longrightarrow \quad C^{-1}AC\mathbf{x} = \lambda\mathbf{x} \quad \Longrightarrow \quad A(C\mathbf{x}) = \lambda(C\mathbf{x}).$$

That is, $C\mathbf{x}$ is an eigenvector of A with eigenvalue λ.

Now we combine similar matrices with generalized eigenvectors. Fortunately, generalized eigenvectors apply to any matrix, not just matrices in Jordan form.

Theorem 20.10 (Jordan)**.** *Every $A \in \mathcal{M}_{n\times n}$ can be decomposed as*

$$A = XJX^{-1}, \tag{10}$$

where J is the Jordan normal form of A, and X has the generalized eigenvectors of A as columns, in the same order as the eigenvalues in J.

Remark 20.11. Let $A \in \mathcal{M}_{n\times n}$ with $J = X^{-1}AX$ in Jordan normal form, and let B be similar to A. That is, there exists some $C \in \mathcal{M}_{n\times n}$ with $B = CAC^{-1}$. It follows that

$$J = X^{-1}AX = X^{-1}(C^{-1}BC)X = (CX)^{-1}B(CX).$$

In other words, B has the same Jordan normal form as A.

Inquiry 20.12: Let $A \in \mathcal{M}_{n\times n}$, and suppose that the eigenvectors of A are all linearly independent.

- How many Jordan blocks does A have? What does this say about the Jordan normal form of A? It will be a ________ matrix.
- Recall the $X\Lambda X^{-1}$ decomposition from Lecture 18. How does this compare with the decomposition of A from Jordan's theorem?

20.3 Exercises

Exercise 20.1. The matrix A below has a single eigenvalue $\lambda = 6$ with algebraic multiplicity 4 and geometric multiplicity 1. Find all of its generalized eigenvectors.

$$A = \begin{bmatrix} 9 & -1 & -1 & -3 \\ -3 & 5 & 1 & 1 \\ 5 & -5 & 5 & -9 \\ 3 & 1 & -1 & 5 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

Exercise 20.2. How many different matrices in $\mathcal{M}_{7\times 7}$, up to similarity, are there with one eigenvalue $\lambda = 2$ that has algebraic multiplicity 2 and

1. geometric multiplicity 2?
2. geometric multiplicity 3?
3. any geometric multiplicity?

Exercise 20.3. Let $J \in \mathcal{M}_{6\times 6}$ be a matrix in Jordan form with two eigenvalues 3 (having algebraic multiplicity 4 and geometric multiplicity 2) and -3 (having algebraic multiplicity 2 and geometric multiplicity 1).

1. How many Jordan blocks will J have? Give the two possibilities for their sizes.

2. Suppose that the Jordan blocks of J all have the same size. Find a matrix B that is similar to J and has no zero entries.

3. For the matrix B from part (2.), find all its generalized eigenvectors.

Lecture 21: Generalizing diagonalizability: Singular values

Chapters 7.1,7.2 in Strang's "Linear Algebra"

- Fact 1: No matter what size A has, AA^T and A^TA are both positive semidefinite and have the same nonzero eigenvalues.
- Fact 2: The SVD contains orthonormal bases of the four fundamental subspaces.

- Skill 1: Compute the rank r approximation to A
- Skill 2: Decompose a non-square matrix A by the SVD

This lecture continues with generalizing diagonalizibility. Instead to having some XJX^{-1} decomposition for a square matrix, as in the previous lecture, we get a decomposition for a matrix of any rectangular size.

21.1 Eigenvalues of symmetric matrices

The word *singular* so far has been used when talking about matrices. A square matrix was seen to be singular if its determinant is zero, and non-singular otherwise. Before we begin with the new concept of *singular*, we make two observations.

Remark 21.1. Let any $A \in \mathcal{M}_{m\times n}$. Then $AA^T \in \mathcal{M}_{m\times m}$ and $A^TA \in \mathcal{M}_{n\times n}$

- both have the same nonzero eigenvalues, not counting algebraic multiplicity;
- both are positive semidefinite.

The first points follows by using the usual eigenvalue-eigenvector equations. Suppose that $(\lambda, \mathbf{u})$ is an eigenpair for AA^T, and the $(\mu, \mathbf{v})$ is an eigenpair for A^TA. Then

$$AA^T\mathbf{u} = \lambda\mathbf{u} \quad \Longrightarrow \quad A^TA(A^T\mathbf{u}) = \lambda(A^T\mathbf{u}), \tag{11}$$

$$A^TA\mathbf{v} = \mu\mathbf{v} \quad \Longrightarrow \quad AA^T(A\mathbf{v}) = \mu(A\mathbf{v}). \tag{12}$$

In other words, we the immediately get that $(\lambda, A^T\mathbf{u})$ is an eigenpair for A^TA and $(\mu, A\mathbf{v})$ is an eigenpair for AA^T. However, we only get this conclusion if $A\mathbf{u}$ and $A^T\mathbf{v}$ are not the zero vector! Recall that an eigenvector cannot be the zero vector. This situation is explored more in Inquiry 21.2.

The second point follows by obsevervation. Let $\mathbf{x} \in \mathbf{R}^m$ and $\mathbf{y} \in \mathbf{R}^n$. Then

$$\mathbf{x}^TAA^T\mathbf{x} = (\mathbf{x}^TA)\cdot(A^T\mathbf{x}) = (A^T\mathbf{x})^T\cdot(A^T\mathbf{x}) = (A^T\mathbf{x})\bullet(A^T\mathbf{x}) = \|A^T\mathbf{x}\|^2 \geqslant 0,$$

$$\mathbf{y}^TA^TA\mathbf{y} = (\mathbf{y}^TA^T)\cdot(A\mathbf{y}) = (A\mathbf{y})^T\cdot(A\mathbf{y}) = (A\mathbf{y})\bullet(A\mathbf{y}) = \|A\mathbf{y}\|^2 \geqslant 0,$$

where the last inequality follows from the nonnegativity of the norm $\|\cdot\|$. Note that $\mathbf{x}^TAA^T\mathbf{x}$ may be equal to zero even when $\mathbf{x} \neq 0$. Indeed, if $A^T\mathbf{x} = 0$, it simply means there is linear dependence among the columns of A^T (equivalently, among the rows of A).

Inquiry 21.2: Consider the matrix $A = \left[\begin{smallmatrix} 1&1&1&2&3\\ 2&2&2&2&3\\ 3&3&3&3&3 \end{smallmatrix}\right] \in \mathcal{M}_{3\times 5}$. You may use a computer for this Inquiry.

- Compute the 3 eigenvalues of AA^T and the 5 eigenvalues of A^TA.
- Compute the eigenvectors for the zero eigenvalues of A^TA are zero.
- Attempt to use Equation (11) to get the associated eigenvectors for AA^T. What is happening?

Definition 21.3: Let $A \in \mathcal{M}_{m\times n}$. The *singular values* of A are the square roots of the eigenvalues that AA^T and A^TA have in common.

Example 21.4. Continuing with the matrix A from Inquiry 21.2, we can apply the decomposition from Remark 19.4. For AA^T, suppose that it has eigensystem $\{(\lambda_1, \mathbf{u}_1), (\lambda_2, \mathbf{u}_2), (\lambda_3, \mathbf{u}_3)\}$, with $\lambda_1 > \lambda_2 > \lambda_3$. Then, using decimals,

$$AA^T \approx \underbrace{83.38}_{\lambda_1} \underbrace{\begin{bmatrix} 0.17 & 0.23 & 0.3 \\ 0.23 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.53 \end{bmatrix}}_{\mathbf{u}_1\mathbf{u}_1^T} + \underbrace{2.49}_{\lambda_2} \underbrace{\begin{bmatrix} 0.69 & 0.08 & -0.45 \\ 0.08 & 0.01 & -0.05 \\ -0.45 & -0.05 & 0.3 \end{bmatrix}}_{\mathbf{u}_2\mathbf{u}_2^T} + \underbrace{0.13}_{\lambda_3} \underbrace{\begin{bmatrix} 0.14 & -0.31 & 0.15 \\ -0.31 & 0.69 & -0.34 \\ 0.15 & -0.34 & 0.17 \end{bmatrix}}_{\mathbf{u}_3\mathbf{u}_3^T}.$$

Notice the very large eigenvalue and the two smaller ones. This decomposition will be useful when we ignore the smaller eigenvalues.

Inquiry 21.2 above showed that if AA^T has more eigenvalues than A^TA, or vice versa, then the extra eigenvalues are zero. However, this does not imply that AA^T and A^TA have the same number of independent eigenvectors!

Remark 21.5. Let $A \in \mathcal{M}_{m\times n}$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbf{R}^m$ be the eigenvectors of AA^T and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbf{R}^n$ be the eigenvectors of A^TA, where both are repeated depending on algebraic multiplicity. Without loss of generality, we assume that $n \geqslant m$, so $\mathbf{v}_{m+1}, \ldots, \mathbf{v}_n$ are all eigenvectors for the zero eigenvalue. Let $\sigma_1, \ldots, \sigma_m \in \mathbf{R}$ be such that

$$AA^T\mathbf{u}_i = \sigma_i^2\mathbf{u}_i \qquad \text{and} \qquad A^TA\mathbf{v}_i = \sigma_i^2\mathbf{v}_i,$$

for all $i = 1, \ldots, m$. We may do this because AA^T and A^TA are both positive semidefinite (so we can take square roots of the eigenvalues). We use σ instead of λ because these are the *singular values* - the letter σ is the letter "s" in Greek. The relationship among the $\mathbf{u}_i$, $\mathbf{v}_i$, σ_i and the original matrix A is then given by

$$A^T\mathbf{u}_i = \sigma_i\mathbf{v}_i \qquad \text{and} \qquad A\mathbf{v}_i = \sigma_i\mathbf{u}_i,$$

as multiplying the left equation by A on the left means the equation on the right must be true (for the previous equation to hold). Combining all the equations $A\mathbf{v}_i = \sigma_i\mathbf{u}_i$ into a big equation, and assuming that the $\mathbf{u}_i$ are orthonormal, and the $\mathbf{v}_i$ are orthonormal as well ,we get the following decomposition:

$$A\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \sigma_1\mathbf{u}_1 & \sigma_2\mathbf{u}_2 & \cdots & \sigma_m\mathbf{u}_m \\ | & | & & | \end{bmatrix}$$

$$A\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_m \end{bmatrix}$$

$$A\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & \mathbf{v}_1 & - \\ - & \mathbf{v}_2 & - \\ & \vdots & \\ - & \mathbf{v}_m & - \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_m \end{bmatrix} \begin{bmatrix} - & \mathbf{v}_1 & - \\ - & \mathbf{v}_2 & - \\ & \vdots & \\ - & \mathbf{v}_m & - \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_m \end{bmatrix} \begin{bmatrix} - & \mathbf{v}_1 & - \\ - & \mathbf{v}_2 & - \\ & \vdots & \\ - & \mathbf{v}_m & - \end{bmatrix} \\ &= \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T + \cdots + \sigma_m\mathbf{u}_m\mathbf{v}_m^T. \end{aligned}$$

Definition 21.6: The *singular value decomposition* of $A \in \mathcal{M}_{m\times n}$ is $A = U\Sigma V^T$, where

- $U \in \mathcal{M}_{m\times m}$ has the eigenvectors of AA^T as columns,
- $V \in \mathcal{M}_{n\times n}$ has the eigenvectors of A^TA as columns,
- $\Sigma \in \mathcal{M}_{m\times n}$ has the singular values of A on the diagonal of its upper left $\text{rank}(A) \times \text{rank}(A)$ submatrix, in decreasing order from the largest in Σ_{11}.

The order of the eigenvectors in U and V corresponds to the order of the singular values in Σ. The vectors $\mathbf{u}_i$ are called the *left singular vectors* and the $\mathbf{v}_i$ are called the *right singular vectors* of A.

Singular value decomposition allows us to have an eigenvalue-eigenvecftor type decomposition for non-square matrices. This is very powerful, as most data in real life is not square.

Inquiry 21.7: Consider the two flags below (of Lithuania and Benin), given as matrices.

$$L = \begin{bmatrix} y & y & y & y & y & y & y & y & y \\ y & y & y & y & y & y & y & y & y \\ g & g & g & g & g & g & g & g & g \\ g & g & g & g & g & g & g & g & g \\ r & r & r & r & r & r & r & r & r \\ r & r & r & r & r & r & r & r & r \end{bmatrix} \qquad B = \begin{bmatrix} g & g & g & y & y & y & y \\ g & g & g & y & y & y & y \\ g & g & g & y & y & y & y \\ g & g & g & r & r & r & r \\ g & g & g & r & r & r & r \\ g & g & g & r & r & r & r \end{bmatrix}$$

- How many singular values do these two matrices have?
- Express both matrices as sums of rank one matrices.

You may use a computer for this task, and a Python function such as `svd` from the package `scipy.linalg`. You will need to convert colors to numbers (the choice of number does not matter, but distinct colors should have distinct numbers).

Very often we do not need the whole decomposition, only a part of it.

Definition 21.8: Let $A \in \mathcal{M}_{m\times n}$, and let $\sigma_1, \sigma_2, \ldots$ be the singular values of A in decreasing order. The *rank r approximation* of A is the sum

$$\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T \in \mathcal{M}_{m\times n},$$

for every $1 \leqslant r \leqslant \text{rank}(A)$. If $r = \text{rank}(A)$, then the rank r approximation of A is equal to A.

These rank r approximations help is massively reduce the amount of "information" in a matrix. For example, given a 100×100 matrix, which has $100^2 = 10\,000$ numbers, we could just consider the rank 5 approximation, which has $5 + 5 \cdot (100 + 100) = 1005$ numbers, an approximately 90% reduction in size.

Inquiry 21.9: This inquiry explores how "similar" the rank r approximations are to the input.

- Open up the Google Colab notebook (link here) and execute the cells in your Python IDE.
- For each image, find the r for which the rank r approximation "essentially looks like" the input image. What percentage reduction in information size did this achieve?
- Find some images on your own, and perform the same steps as above. Without using single color images, try to find the images that have the highest recduction in size.

21.2 Bases in the decomposition

For this section, let $r = \text{rank}(A) \leqslant \min\{m, n\}$, for $A \in \mathcal{M}_{m\times n}$. We have alrerady seen the decomposition of A into three matrices, using eigenvalues and eigenvectors of AA^T and A^TA:

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_r \\ | & | & & | \end{bmatrix}}_{\text{eigenvectors of } AA^T} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}}_{\text{singular values}} \underbrace{\begin{bmatrix} - & \mathbf{v}_1^T & - \\ - & \mathbf{v}_2^T & - \\ & \vdots & \\ - & \mathbf{v}_r^T & - \end{bmatrix}}_{\text{eigenvectors of } A^TA} . \tag{13}$$

Hding in this equation are the bases for the *four fundamental subspaces* that we have already seen in Lecture 8.

Remark 21.10. The rank(A)-approximation of A contains orthonormal basis vectors of other subspaces:

$$A = \Bigg[\underbrace{\begin{matrix} | & & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_r \\ | & & | \end{matrix}}_{\text{column space}} \;\; \underbrace{\begin{matrix} | & & | \\ \mathbf{u}_{r+1} & \cdots & \mathbf{u}_m \\ | & & | \end{matrix}}_{\text{left nullspace}}\Bigg] \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ & 0 & & 0 \end{bmatrix}}_{\substack{m-r \text{ rows,} \\ n-r \text{ columns}}} \begin{bmatrix} - & \mathbf{v}_1^T & - \\ & \vdots & \\ - & \mathbf{v}_r^T & - \\ - & \mathbf{v}_{r+1}^T & - \\ & \vdots & \\ - & \mathbf{v}_n & - \end{bmatrix} \begin{matrix} \text{row space} \\ \\ \text{nullspace} \end{matrix}$$

Example 21.11. Let's compute the full SVD for a matrix, and get the appropriate bases. Consider

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ -1 & -1 \end{bmatrix}, \qquad AA^T = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 8 & -4 \\ -2 & -4 & 2 \end{bmatrix}, \qquad A^TA = \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}.$$

It is immediate that A has rank 1, as the rows are all multiples of the first row. We already know both A^TA and AA^T have the same eigenvalues, so we just find them for the easier of the two, A^TA. The roots of the characteristic polynomial are found by

$$0 = \det(A^TA - \lambda I) = (6-\lambda)^2 - 36 = 36 - 12\lambda + \lambda^2 - 36 = \lambda^2 - 12\lambda = (\lambda - 12)\lambda,$$

so the eigenvalues are 12 and 0. Hence the only singular value is $\sigma_1 = 2\sqrt{3}$. To find the eigenvectors, we row reduce the appropriate augmented matrices, remembering to normalize the eigenvectors.

$$12 \text{ for } AA^T : \begin{bmatrix} -10 & 4 & -2 & 0 \\ 4 & -4 & -4 & 0 \\ -2 & -4 & -10 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$0 \text{ for } AA^T : \begin{bmatrix} 2 & 4 & -2 & 0 \\ 4 & 8 & -4 & -0 \\ -2 & -4 & 2 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$12 \text{ for } A^TA : \begin{bmatrix} -6 & 6 & 0 \\ 6 & -6 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$0 \text{ for } A^TA : \begin{bmatrix} 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

We could have also found $\mathbf{v}_1$ by $A^T\mathbf{u}_1 = 2\sqrt{3}\mathbf{v}_1$. This gives us the complete decomposition

$$A = \underbrace{\begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{2} & -2/\sqrt{5} \\ -2/\sqrt{6} & 0 & 1/\sqrt{5} \\ 1/\sqrt{6} & 1/\sqrt{2} & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T},$$

as well as bases

$$\text{col}(A) = \text{span}\left\{ \begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}, \qquad \text{null}(A^T) = \text{span}\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} \right\},$$

$$\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}, \qquad \text{null}(A) = \text{span}\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}.$$

Remark 21.12. If A is symmetric, then the SVD is the same as the $Q\Lambda Q^T$-decomposition. In this way, the SVD is a more general decomposition that captures the nice properties of the $Q\Lambda Q^T$-decomposition.

21.3 Exercises

Exercise 21.1. Consider the two "matrices" below.

$$L = \begin{bmatrix} r & r & r & r & r & r & r & r & r & r \\ r & r & r & r & r & r & r & r & r & r \\ w & w & w & w & w & w & w & w & w & w \\ r & r & r & r & r & r & r & r & r & r \\ r & r & r & r & r & r & r & r & r & r \end{bmatrix} \qquad B = \begin{bmatrix} w & w & w & r & w & r & w & w & w \\ w & w & w & r & w & r & w & w & w \\ r & r & r & r & w & r & r & r & r \\ w & w & w & w & w & w & w & w & w \\ r & r & r & r & w & r & r & r & r \\ w & w & w & r & w & r & w & w & w \\ w & w & w & r & w & r & w & w & w \end{bmatrix}$$

1. Express L, the flag of Latvia, as a rank one product of two vectors.
2. Express B, the flag of Latvian battleships, as a sum of two rank one matrices. That is, decompose B as $B = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T$.

Exercise 21.2. This question uses Python. You may use the folowing resources:

- Sample code: `jlazovskis.com/teaching/linearalgebra`
- Sample images: `links.uwaterloo.ca/Repository.html`

Find a grayscale image online at least 100×100 pixels in size. It does not have to be square.

$\bowtie$ 1. Find the singular values of the image. How many of them are less than 1/100 of the largest singular value?

$\bowtie$ 2. Compute the rank r approximation to the image for $r = 1, 2, 3, 5, 10$.

3. If the image had size $m \times n$, what is the percent reduction in size for the rank r approxmation?

Exercise 21.3. Let $a \in \mathbf{R}_{\neq 0}$, and consider the matrix

$$A = \begin{bmatrix} a & 0 & a & 0 \\ 0 & 0 & 0 & 2a \end{bmatrix}.$$

1. Compute the SVD of A by finding the eigenvalue / eigenvector pairs for AA^T and A^TA.

2. What are the dimensions of the four fundamental subspaces of A?

Exercise 21.4. 1. Construct a 3×4 matrix with singular values $1, 2, 3$.

2. Construct a 2×2 rank 1 matrix with right singular vectors $\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}, \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$.

3. Find the rank 1 and rank 2 approximations for

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hint: Since two eigenvalues are the same, there are two rank 2 approximations!

Part IV

Generalizations and applications

Lecture 22: Principal component analysis

Chapter 7.3 in Strang's "Linear Algebra"

- Fact 1: The first principal component solves the perpendicular least squares problem
- Fact 2: The first two principal components give a reasonable way to plot high-dimensional data

- Skill 1: Solve the perpendicular least squares problem using SVD
- Skill 2: Identify the principal components of $A \in \mathcal{M}_{m\times n}$, in terms of the total covariance of A
- Skill 3: Normalize and center data on its mean

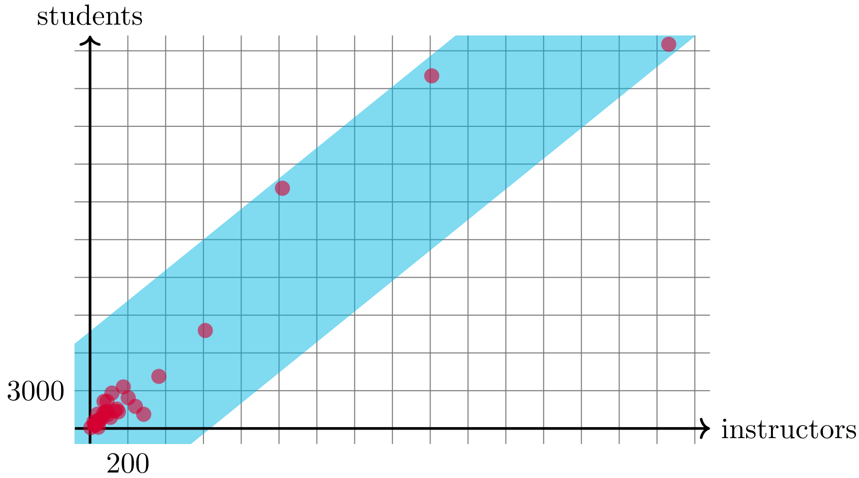
In the previous lecture, we saw how to simplify images, thought of as a matrix A, for compressed communication, using the eigenvectors of AA^T and A^TA, which appear in the singular value decomposition of A. in this lecture we will apply SVD, but to a different problem: *dimensionality reduction*.

22.1 The first significant direction of data

All data used in this lecture is available on the course website `jlazovskis.com/teaching/linearalgebra`.

Example 22.1. Consider the following data set, representing the number of instructors (x-value) and the number of students (y-value) at 32 different post-secondary institutions in Latvia.

$$\{(1531, 15260), \ldots, (2, 33)\}$$



There seems to be a general trend! In Lecture 11 we saw how to approximate this data with a least squares line of best fit. We do something similar now, but slightly differently, and as motivation for higher dimensions. Each pair in this data set is a *sample*, so we can costruct a *sample matrix* $A \in \mathcal{M}_{2\times 32}$.

Definition 22.2: Let $A \in \mathcal{M}_{m\times n}$ and consider each of the n columns of A as a sample. There are two matrices associated to A:

$$M_{ij} = A_{ij} - \underbrace{\frac{1}{n}\sum_{k=1}^{n} A_{ik}}_{\text{mean of row } i}\ , \qquad S = \frac{MM^T}{n-1}.$$

A has a *mean-centered* matrix $M \in \mathcal{M}_{m\times n}$ and a *sample covariance* matrix $S \in \mathcal{M}_{m\times m}$.

By definition, S is symmetric.

Inquiry 22.3: Consider the matrix $A = \left[\begin{smallmatrix} 2 & 3 & -1 & 6 & 1 \\ 0 & -3 & -4 & 1 & -1 \end{smallmatrix}\right]$.

- Compute the mean-centered matrix M.
- Suppose you add one column (sample) to M. Will M still be mean-centered? Why or why not?
- Suppose you add two columns to M. What must be true about the two columns for the new M to still be mean-centered?

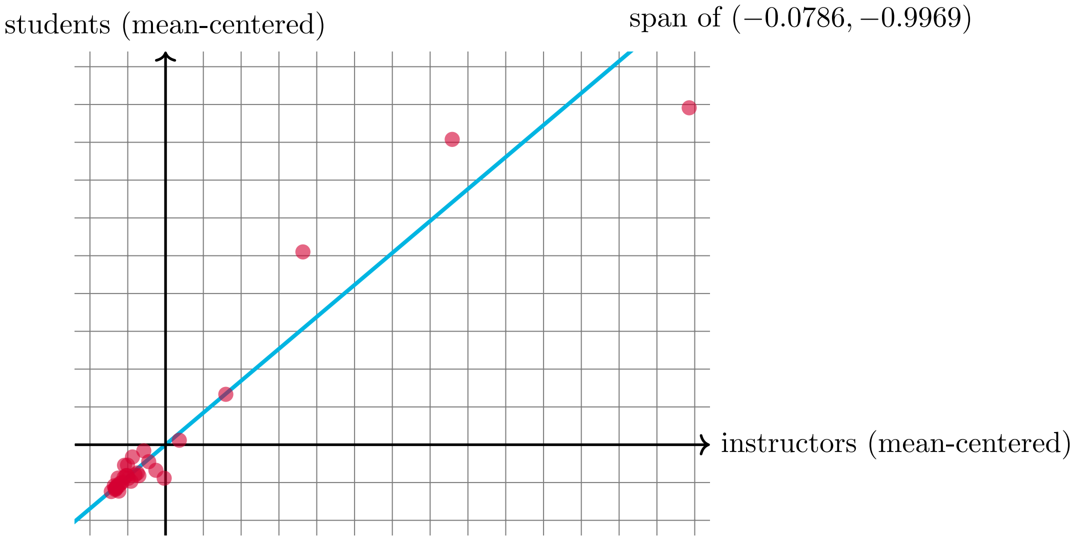
Continuing with Example 22.1, we find the means and center the matrix accordingly:

mean of row 1 (students):	145.6
mean of row 2 (instructors):	1890.6

This lets us create the mean-centered 2×32 matrix M and the sample covariance 2×2 matrix S for the data. The key lies in the singular value decomposition of

$$S = \begin{bmatrix} 73909.14 & 864786.84 \\ 864786.84 & 10971745.39 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.0786 & -0.9969 \\ -0.9969 & 0.0786 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 11039942.91 & 0 \\ 0 & 5711.62 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} -0.0786 & -0.9969 \\ -0.9969 & 0.0786 \end{bmatrix}}_{V^T}.$$

Since S is symmetric, the matrices U and V are the same. The singular vector with the largest eigenvalue identifies the *principal component* of the mean-centered data. This can be thought of as a 1-dimensional subspace of $\mathbf{R}^m$ that does the best job (that a 1-dimensional subspace could do) of approximating all the data. The first eigenvalue dominates the second one, indicating the data is very close to a straight line. The straight line is given by the eigenvector corresponding to the large eigenvalue.



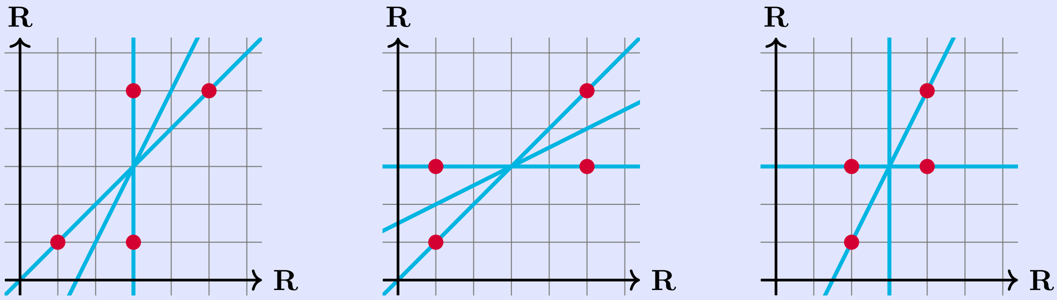
The line here is $y = \frac{0.9969}{0.0786}x$, which best approximates the mean-centered data. The line that best approximates the original data is this line, but shifted back by the mean:

$$y = \frac{0.9969}{0.0786}(x - 145.6) + 1890.6.$$

Definition 22.4: Let $A \in \mathcal{M}_{m\times n}$. The *(first) principal component* of A is the singular vector corresponding to the largest singular value of A.

The first principal component of A solves the *perpendicular least squares* problem. That is, the first eigenvector minimizes the square of the distance from its line to the data. This is alternative to the least squares solution we saw in Lecture 10, which minimized the the vertical distance.

Inquiry 22.5: Consider the data sets and lines below.



- For each of the grids above, indicate which of the three lines you think corresponds to the linear least squares approximation and which corresponds to the first principal component.
- Check your answers by computing the least squares linear approximation and the first principal component to the data sets. Use the interactive plot (link here).
- Which approximation do you think is better? Why?
- Try to come up with data for which the difference between the two lines is as big as posible.

The key idea for this inquiry is that least squares minimizes vertical distance and the first principal component minimizes perpendicular distance. "Distance" means the sum of the lengths from each point to the line.

22.2 PCA for higher dimensions

So far we saw data with two coordinates, but very often the data we see is many-dimensional, and has more than one important component. Now we analyze the principal components (that is, singular vectors) corresponding to the several largest singular values.

Example 22.6. The data from Example 22.1 can be augmented with extra data about the change in student and instructor numbers from the previous year. This gives 4-dimensional data, which can not be easily visualized on a page.

iestade	akad_pers_2019	akad_pers_2020	stud_2019	stud_2020
Latvijas Universitāte	1182	1531	15250	15260
Rīgas Tehniskā universitāte	930	904	14383	14006
Daugavpils Universitāte	194	182	2163	2068
⋮	⋮	⋮	⋮	⋮
Latvijas Nacionālā aizsardzības akadēmija	10	10	269	262

If we want to consider the change (percent), then we need to normalize the data, to make sure that a change in every coordinate is taken into account similarly.

Definition 22.7: Let $\mathbf{x} \in \mathbf{R}^n$. The *normalization* of $\mathbf{x}$ is a vector $\hat{\mathbf{x}} \in \mathbf{R}^n$ that is either:

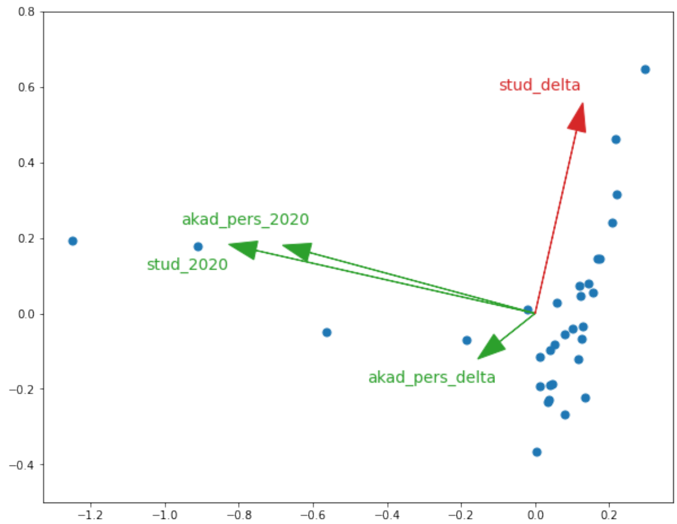
- a multiple of $\mathbf{x}$ so that it has unit length: $\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$
- a shift and scale the vector so that it lies in $[0,1]^n$: $\hat{\mathbf{x}} = \frac{\mathbf{x}-\mathbf{m}}{M-m}$, where $m = \min_i x_i$, $M = \max_i x_i$, and $\mathbf{m} = [m\ m\ \cdots\ m]^T$.

The second case is also called *min-max normalization*, and is the normalization used here.

We normalize each row, then center it at zero, then compute the sample covariance matrix, and finally get its SVD. The matrices U and Σ from the SVD are below.

$$U = \begin{bmatrix} -0.61 & 0.161 & -0.011 & -0.776 \\ -0.754 & 0.167 & -0.09 & 0.629 \\ -0.096 & -0.075 & 0.992 & 0.046 \\ -0.224 & -0.97 & -0.094 & -0.025 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 0.802 & 0 & 0 & 0 \\ 0 & 0.363 & 0 & 0 \\ 0 & 0 & 0.164 & 0 \\ 0 & 0 & 0 & 0.0145 \end{bmatrix}$$



Looking at the first two columns of U (the first two singular vectors), we see that the second coordinate (student number) has the largest magnitude for the first singular vector $\mathbf{u}_1$, and the last coordinate (change in student number) has the largest magnitude for the second singular vector $\mathbf{u}_2$:

$$\mathbf{x}_{\text{new}} = \text{proj}_{\text{span}(\mathbf{u}_1,\mathbf{u}_2)}(\mathbf{x}_{\text{old}}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x_1 = \frac{\mathbf{u}_1 \bullet \mathbf{x}_{\text{old}}}{\mathbf{u}_1 \bullet \mathbf{u}_1}, \quad x_2 = \frac{\mathbf{u}_2 \bullet \mathbf{x}_{\text{old}}}{\mathbf{u}_2 \bullet \mathbf{u}_2}.$$

The first two singular vectors are the "defining directions" of the data.

Definition 22.8: Let $A \in \mathcal{M}_{m\times n}$, considered as n samples in m coordinates.

- For each $1 \leqslant i \leqslant m$, the *variance* of coordinate i is S_{ii}.

A large variance means coordinate i is spread out, and a small variance means coordinate i is densely packed.

- For each $1 \leqslant i, j \leqslant m$, the *covariance* of coordinate i with coordinate j is $S_{ij} = S_{ji}$.

A large positive covariance means coordinate i increases when coordinate j increases, and a large negative covariance means coordinate i decreases when coordinate j increases.

- The *total variance* of A is $\text{trace}(S)$.

The variance of the data from Example 22.1 is either $\text{trace}(S) = S_{11} + S_{22}$ or $\text{trace}(\Sigma) = \Sigma_{11} + \Sigma_{22}$, since the sum of the eigenvalues of a matrix is the trace of the matrix. The singular value of the first principal component accounts for $\sigma_1/\text{trace}(S) \approx 0.99$, or about 99% of the total covariance. In general, it may take more than the first principal component to accont for so much of the covariance - your choice of when to stop determines the *princial components* of the data.

22.3 Exercises

Exercise 22.1. This question is about the 4 point interactive found on the course website (link here).

1. Create an arrangement of the points with the largest angle possible between the two approximations that you can find. Do you think any angle is possible? Justify your reasoning.

2. Create an arrangement of the points with the largest difference between the sums of the distances that you can find. Besides all points being on a line, what situations give the same sums of distances?

Exercise 22.2. Find samples of high-dimensional (at least 4) data online.

1. Construct the sample covariance matrix S and find the two largest eigenvalue / eigenvector pairs from its SVD.

2. What percentage of the total covariance do the first two principal components cover?

3. Plot the data on the axes of the two principal components.

4. Create two plots of the data having for axes:

   (a) the first principal component against the coordinate with the highest (in magnitude) association

   (b) the second principal component against the coordinate with the highest (in magnitude) association

Exercise 22.3. Create a matrix of 2-dimensional data for which the first principal component of the data is a multiple of the eigenvector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, for $a, b \in \mathbf{R}_{\neq 0}$. Make sure that:

- the matrix has at least 3 columns (samples),

- no 3 samples are colinear.

Exercise 22.4.

1. Create a matrix of 3-dimensional data for which first two principal components are the vectors $[\,1\;0\;0\,]^T$ and $[\,0\;1\;0\,]^T$. Make sure that:

   - the data is centered at 0,
   - the matrix has at least 4 columns (samples),
   - no 3 samples are colinear.

2. Do the same as in part (a), but change the last condition to "no 4 samples lie on a plane."

Lecture 23: Linear transformations

Chapters 8.1, 8.2 in Strang's "Linear Algebra"

- Fact 1: A linear transformation is the same thing as a matrix.
- Fact 2: A linear transformtion is injective iff it is surjective.

- Skill 1: Determine whether or not a function is a linear transformation.
- Skill 2: Construct a matrix for a linear transformation, given what it does to a basis.
- Skill 3: Construct the image and kernel of a linear transformation

This lecture focuses on a generalization: the connection between $m \times n$ matrices and functions $\mathbf{R}^n \to \mathbf{R}^m$. We have already seen the interpretation of a matrix as a function with the rotation matrix R_θ in Lecture 9. By the end of this lecture, we will see that every such function comes from a matrix.

23.1 Types of linear transformations

Definition 23.1: Let V, W be vector spaces. A *linear transformation*, or *linear map*, is a function $f \colon V \to W$ that satisfies

$$f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}) \qquad \text{and} \qquad f(c\mathbf{x}) = cf(\mathbf{x}) \tag{14}$$

for every $\mathbf{x}, \mathbf{y} \in V$ and every $c \in \mathbf{R}$. These are conditions for *linearity*.

Example 23.2. We have already seen examples (and non-examples) of linear transformations:

- Every $m \times n$ matrix is a linear transformation $\mathbf{R}^n \to \mathbf{R}^m$, because $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.
- The shift function $\mathbf{x} \mapsto \mathbf{x}+\mathbf{y}$ for nonzero $\mathbf{y}$ is not linear, because splitting up the function on two vectors adds $2\mathbf{y}$ instead of just $\mathbf{y}$.
- The length function is not a linear transformation $\mathbf{R}^n \to \mathbf{R}$, because

$$\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\| = \sqrt{3}, \qquad \text{but} \qquad \left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\| = 0 \neq 2\sqrt{3}.$$

Inquiry 23.3: Each of the functions below are linear. For each, show that the two conditions for linearity are satisfied.

- the dot product of a vector $\mathbf{v} \in \mathbf{R}^3$ with $\left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right] \in \mathbf{R}^3$, as a function $\mathbf{R}^3 \to \mathbf{R}$
- projection of a vector $\mathbf{v} \in \mathbf{R}^3$ to the x-axis, considered as the span of $\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$
- differentiation and integration on the space $C[\mathbf{R}]$ of continuous functions

Each of the functions below is not linear. For each, show which of the linearity conditions are violated.

- addition of the vector $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$: $f(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]) = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] + \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$
- squaring of every component: $f(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]) = \left[\begin{smallmatrix} x^2 \\ y^2 \end{smallmatrix}\right]$

Linearity is good because it gives a complete picture with a small amount of information.

Proposition 23.4. Any linear map $V \to W$ is completely determined by what it does to the basis of V.

This follows immediately by linearity. Another way to say the above proposition is that choosing a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of V and taking any (not necessarily linearly independent!) vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n \in W$, there is only one unique linear map $f \colon V \to W$ for which $f(\mathbf{v}_i) = \mathbf{w}_i$, for all i.

Inquiry 23.5: This inquiry is about the vector space $\mathbf{R}^3$.

- Come up with two different bases $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for $\mathbf{R}^3$.
- Let A be the 2×3 matrix with $A\mathbf{v}_1 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $A\mathbf{v}_2 = \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$ and $A\mathbf{v}_3 = \left[\begin{smallmatrix}-1\\2\end{smallmatrix}\right]$. What is A?
- Let B be the 4×3 matrix with $B\mathbf{w}_1 = \left[\begin{smallmatrix}1\\0\\0\\1\end{smallmatrix}\right]$, $B\mathbf{v}_2 = \left[\begin{smallmatrix}0\\2\\0\\2\end{smallmatrix}\right]$ and $B\mathbf{v}_3 = \left[\begin{smallmatrix}0\\0\\-1\\2\end{smallmatrix}\right]$. What is B?
- What are the ranks of A and B? How are these numbers related to the dimension of $\mathbf{R}^3$?

Every linear transformation $V \to W$ creates new subspaces of V and of W.

Definition 23.6: Let $f \colon V \to W$ be a linear transformation.

- The *kernel* of f is $\ker(f) = \{\mathbf{x} \in V \ : \ f(\mathbf{x}) = 0\} \subseteq V$
- The *image*, or *range* of f is $\operatorname{im}(f) = \{f(\mathbf{x}) \in W \ : \ \mathbf{x} \in\} \subseteq W$

Note that $\ker(f) \subseteq V$ is a subspace of V, and $\operatorname{im}(f) \subseteq W$ is a subspace of W.

Example 23.7. For $f(\mathbf{x}) = A\mathbf{x}$, multiplication by a matrix, the kernel is the nullspace and the image is the column space. That is,

$$\ker(f) = \operatorname{null}(A), \qquad \operatorname{im}(f) = \operatorname{col}(A).$$

Recall that a function $f \colon X \to Y$ is *injective*, or *one-to-one*, if $f(a) = f(b)$ implies $a = b$. Further, the function f is *surjective*, or *onto*, if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$. We will apply these concepts to linear transformations.

Proposition 23.8. Let $f \colon V \to W$ be linear.

- f is injective iff $\ker(f) = \{0\}$
- if $\dim(W) = \dim(\operatorname{im}(f))$, then f is surjective.

Inquiry 23.9: This inquiry describes the justification for Proposition 23.8.

- Suppose that $\ker(f) = \{0\}$. Show that assuming $f(\mathbf{x}) = f(\mathbf{y})$ implies that $\mathbf{x} = \mathbf{y}$.
- Suppose that f is injective. Use the second linarity condition with $c = 0$ to show that assuming a nonzero vector is in the kernel of f implies a contradiction.
- Revisit Remark 7.17 and explain why it justifies the second point of the proposition.

Combining injective and surjective linear transformations gives us a very special transformation.

Definition 23.10: A linear transformation $f \colon V \to W$ that is both injective and surjective is an *isomorphism*.

You may have seen the word *bijective* be used for functions that are both injective and surjective, but for linear maps we use this special word. Isomorphisms are important because they preserve the fundamental structure of the vector space V.

Example 23.11. We have already seen examples of isomorphisms:

- The map $f: \mathbf{R}^n \to \mathbf{R}^n$ with $f(\mathbf{x}) = 2\mathbf{x}$ is an isomoprhism.
- The change of basis matrix from Lecture 7 is an isomorphism
- The dot product of any vector in $\mathbf{R}^2$ with $(-1, 2)$ is not an isomoprhism, as it fails injectivity: $(3, 4) \cdot (-1, 2) = (-5, 0) \cdot (-1, 2)$.

23.2 The matrix of a linear transformation

Theorem 23.12. *Let $f: \mathbf{R}^n \to \mathbf{R}^m$ be linear. Then there is a unique matrix A for which $A\boldsymbol{x} = f(\boldsymbol{x})$ for all $x \in \mathbf{R}^n$.*

Proof. First we do this proof in a special case, using the standard bases $\mathbf{e}_1, \ldots, \mathbf{e}_n$ for $\mathbf{R}^n$ and $\mathbf{e}_1, \ldots, \mathbf{e}_m$ for $\mathbf{R}^m$. By Proposition 23.4, f is completely determined by what it does on the $\mathbf{e}_i$. Suppose that

$$\begin{aligned} f(\mathbf{e}_1) &= a_{11}\mathbf{e}_1 + \cdots + a_{m1}\mathbf{e}_m, \\ f(\mathbf{e}_2) &= a_{12}\mathbf{e}_1 + \cdots + a_{m2}\mathbf{e}_m, \\ &\vdots \\ f(\mathbf{e}_n) &= a_{1n}\mathbf{e}_1 + \cdots + a_{mn}\mathbf{e}_m, \end{aligned}$$

for some $a_{ij} \in \mathbf{R}$. Then on an arbitrary $\mathbf{x} = b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n \in \mathbf{R}^n$, the linear map f takes it to

$$\begin{aligned} f(\mathbf{x}) &= f(b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n) \\ &= b_1 f(\mathbf{e}_1) + \cdots + b_n f(\mathbf{e}_n) \\ &= b_1(a_{11}\mathbf{e}_1 + \cdots + a_{m1}\mathbf{e}_m) + \cdots + b_n(a_{1n}\mathbf{e}_1 + \cdots + a_{mn}\mathbf{e}_m) \\ &= (b_1a_{11} + \cdots + b_na_{1n})\mathbf{e}_1 + \cdots + (b_1a_{m1} + \cdots b_na_{mn})\mathbf{e}_m. \end{aligned}$$

Since $\mathbf{e}_i$ is all zeros except a 1 on line i, the last line above can be rewritten as

$$\begin{bmatrix} b_1a_{11} + \cdots + b_na_{1n} \\ b_2a_{21} + \cdots + b_na_{2n} \\ \vdots \\ b_1a_{m1} + \cdots b_na_{mn} \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = A(b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n) = A\mathbf{x}.$$

So in this case, f is exactly A.

In the general case, where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is some basis for $\mathbf{R}^n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ is some basis for $\mathbf{R}^m$, construct the change of basis matrices C_V, that takes the v_i to the $\mathbf{e}_i$, and C_W, that takes the $\mathbf{w}_i$ to the e_i. Then the matrix of the function f is $C_W^{-1}AC_V$. □

Inquiry 23.13: This inquiry connects linear transformations with matrices. Recall that A^T is the transpose of A.

- Considering the transpose as a "function" $\mathcal{M}_{3\times 2} \to \mathcal{M}_{2\times 3}$, explain why this cannot be linear.
  Hint: How would this work as a matrix multiplication?

- Explain why the function $f\colon \mathcal{M}_{3\times 2} \to \mathbf{R}^6$, for which

$$f\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & a_{21} & a_{22} & a_{31} & a_{32} \end{bmatrix}^T$$

  is linear. What is its inverse, and is it linear as well?

- Describe a function $g\colon \mathbf{R}^6 \to \mathbf{R}^6$ so that

$$(f^{-1} \circ g \circ f)\colon \mathcal{M}_{3\times 2} \to \mathcal{M}_{2\times 3}$$

  produces the transpose of a matrix. Is it linear?

This Theorem above has several implications. Combining the rank-nullity theorem from Lecture 8 along with observations above, we immediately get the following.

Corollary 23.14. Let $f\colon V \to W$ be linear, with $\dim(V) = \dim(W)$.

- [Dimension Theorem] $\dim(V) = \dim(\ker(f)) + \dim(\operatorname{im}(f))$
- The map f is surjective iff it is injective

Proof. The first point follows by the rank-nullity theorem and applying Theorem 23.12 in Example 23.7 to describe every linear map as a matrix.

The second point follows immediately from the first point and Proposition 23.8. □

Remark 23.15. We also get a nice result for compositions of linear maps. Given two linear maps $f\colon V \to W$ and $g\colon W \to Z$, their *composition* is a linear map $(g \circ f)\colon V \to Z$ (you will check this in an exercise). If f, g have associated matrices A, B, respectively, then the composition $g \circ f$ has associated matrix BA. This follows by using the equations $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{y}) = B(\mathbf{y})$ in simplifying

$$(g \circ f)(\mathbf{x}) = g(f(\mathbf{x})) = g(A\mathbf{x}) = B(A\mathbf{x}) = (BA)\mathbf{x}.$$

23.3 Exercises

Exercise 23.1. For this question, the vector $T_i(\mathbf{x})$ is simply written $T_i\mathbf{x}$, to both ease notation and as a reminder that linear transformatons are simply matrices. You are given the following transformations T_i:

$$T_1\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} w \\ y \\ z \\ x \end{bmatrix} \qquad T_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2e^y \\ x \end{bmatrix} \qquad T_3\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \qquad T_4\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sin(x^2+y^2) \\ \cos(x^2+y^2) \end{bmatrix}$$

$$T_5\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y+x \\ 0 \\ x^2-y \end{bmatrix} \qquad T_6\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad T_7\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3x \\ z+y \end{bmatrix} \qquad T_8\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+2y \\ y+z \\ 0 \end{bmatrix}$$

1. Which of the T_i are linear? For those that are not, give a counterexample in which one of the linearity conditions fail.

2. Let $S\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation for which

$$ST_5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad ST_8 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad ST_8 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Construct the 3×3 matrix of S.

Exercise 23.2. Prove the claim from Definition 23.6 that the kernel and image of $f\colon V \to W$ are subspaces of V and W, respectively. Use linearity to check the vector space conditions.

Exercise 23.3. Let $f\colon V \to W$ be a linear transformation, and let $v_1, \dots, v_n$ be a basis of V. Show that f is injective iff the set of vectors $f(\mathbf{v}_1), \dots, f(\mathbf{v}_n) \subseteq W$ is linearly independent.

Exercise 23.4. Consider the three orthogonal vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}.$$

1. Find the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$.

2. Construct a symmetric matrix A of full rank for which the unit vectors from part (a) are eigenvectors.

3. Let $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation for which

$$f(\mathbf{x}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad f(\mathbf{y}) = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \qquad f(\mathbf{z}) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Construct the 3×3 matrix for f.

Exercise 23.5. Let V be the vector space of polynomials in two variables x and y of degree at most 2. This space has dimension 6, and has basis with basis $1, x, y, x^2, y^2, xy$. Let $L\colon V \to V$ be the linear transformation defined by $L(f(x, y)) = f(x - y, y - x)$.

1. Find the matrix of L using the basis specified.

2. Find a basis for the image and kernel of L.

Exercise 23.6. Prove the claim from Remark 23.15 that the composition of two linear maps is linear.

Lecture 24: Complex numbers and complex matrices

Chapters 9.1 and 9.2 in Strang

- Fact 1: All the math we have done so far can be considered over $\mathbf{C}$ instead of $\mathbf{R}$
- Fact 2: Complex number adition and multiplication have geomtric meaning

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- Skill 1: Express a complex number in one of four different ways
- Skill 2: Apply the new results for Hermitian vectors and matrices

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In this lecture we will take some time to introduce fully the topic of coplex numbers. The goal is to get a better feel for them and to set the stage for the future topic of *Fourier transforms*. Fortunately, almost all the results we have seen so far with matrices over $\mathbf{R}$ apply to matrices over $\mathbf{C}$ as well.

24.1 The space of complex numbers

Definition 24.1: The *complex numbers* are elements of the set $\mathbf{C} = \{x + iy \ : \ x, y \in \mathbf{R}\}$. The symbol i is the *imaginary number*, having the property that $i^2 = -1$.

- The *standard form* of a complex number is $x + iy$.
- In *Cartesian*, or *rectangular* coordinates this number is written (x, y).

The *real part* of $x + iy$ is x and its *imaginary part* is y.

If $x = 0$, then $z = x + iy$ is a *purely imaginary number*.

Let $z = x + iy$ and $w = a + ib$ be complex numbers and $c \in \mathbf{R}$. Complex number addition and multiplication, and real number multiplication are defined in the following way:

$$\begin{aligned} z + w &= (a + x) + i(y + b) \\ zw &= xa + ixb + iya + i^2tb = (xa - yb) + i(xb + ya) \\ cz &= cx + icy \end{aligned}$$

Inquiry 24.2: The set $\mathbf{C}$ along with complex number addition and scalar multiplication as above form a vector space.

- Show that the function $f\colon \mathbf{C} \to \mathbf{R}^2$, given by $f(x + iy) = (x, y)$ is bijective.
- Give a bijection between $\mathbf{C}^n$ and $\mathbf{R}^{2n}$, for any $n \in \mathbf{N}$.
- Let $z = 2 + i \in \mathbf{C}$. Is the function $m\colon \mathbf{C} \to \mathbf{C}$, given by $m(x + iy) = (2 + i)(x + iy)$ a linear transformation?

Example 24.3. What does the complex number $(1 + i)^{-2}$ look like in standard form? Observe that

$$\frac{1}{(1+i)^2} = \frac{1}{1 + 2i + i^2} = \frac{1}{1 + 2i - 1} = \frac{1}{2i} = \frac{1}{2i}\frac{i}{i} = \frac{i}{-2i} = \frac{-1}{2}i.$$

Definition 24.4: Let $z = x + yi \in \mathbf{C}$. The (*complex*) *conjugate* of z is $\overline{z} = z^* = x - iy$. The *absolute value*, or *modulus* of z is

$$|z| = \sqrt{z\overline{z}} = \sqrt{(x + iy)(x - iy)} = \sqrt{x^2 + y^2}.$$

Proposition 24.5. Let $z = x + iy, w = a + ib \in \mathbf{C}$. Then the conjugate satisfies:

1. $\overline{z+w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z}\,\overline{w}$
3. $\overline{\overline{z}} = z$
4. $z + \overline{z} = 2x$
5. $z - \overline{z} = 2yi$
6. $z^{-1} = \overline{z}/|z|^2$ for $z \neq 0$

And the absolute value satisfies:

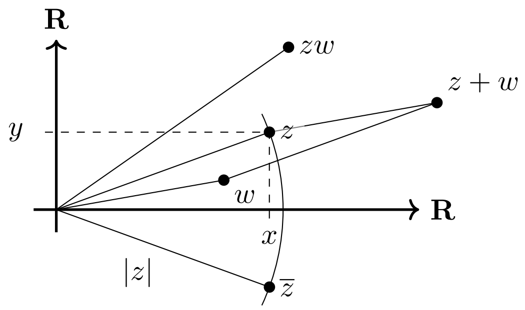
1. $|z| = 0$ iff $z = 0$
2. $|\overline{z}| = |z|$
3. $|zw| = |z||w|$
4. $|z + w| \leqslant |z| + |w|$

Definition 24.6: The third way to express $z = x+iy \in \mathbf{C}$ is with *polar coordinates* (r, θ), where $r = |z|$ and θ is the angle from the positive x axis to the vector (x, y). Note that

$$x + iy = r\cos(\theta) + ir\sin(\theta) = re^{i\theta},$$

where the second equality is known as *Euler's formula*. This last expression is in *exponential form*.

Remark 24.7. All that we have seen so far about the complex numbers, and a new observation about multiplying complex numbers, can be drawn together in a picture.



$$zw = r_z r_w e^{i(\theta_z + \theta_w)}$$

$$z + w = r_w \cos(\theta_w) + ir_w \sin(\theta_w)$$

$$z = x + iy = r_z \cos(\theta_z) + ir_z \sin(\theta_z)$$

$$\overline{z} = x - iy = r_z \cos(\theta_z) - ir_z \sin(\theta_z)$$

Remark 24.8. Putting complex numbers into polar coordinates makes computations in standard form much easier. For $z = re^{i\theta}$, we have:

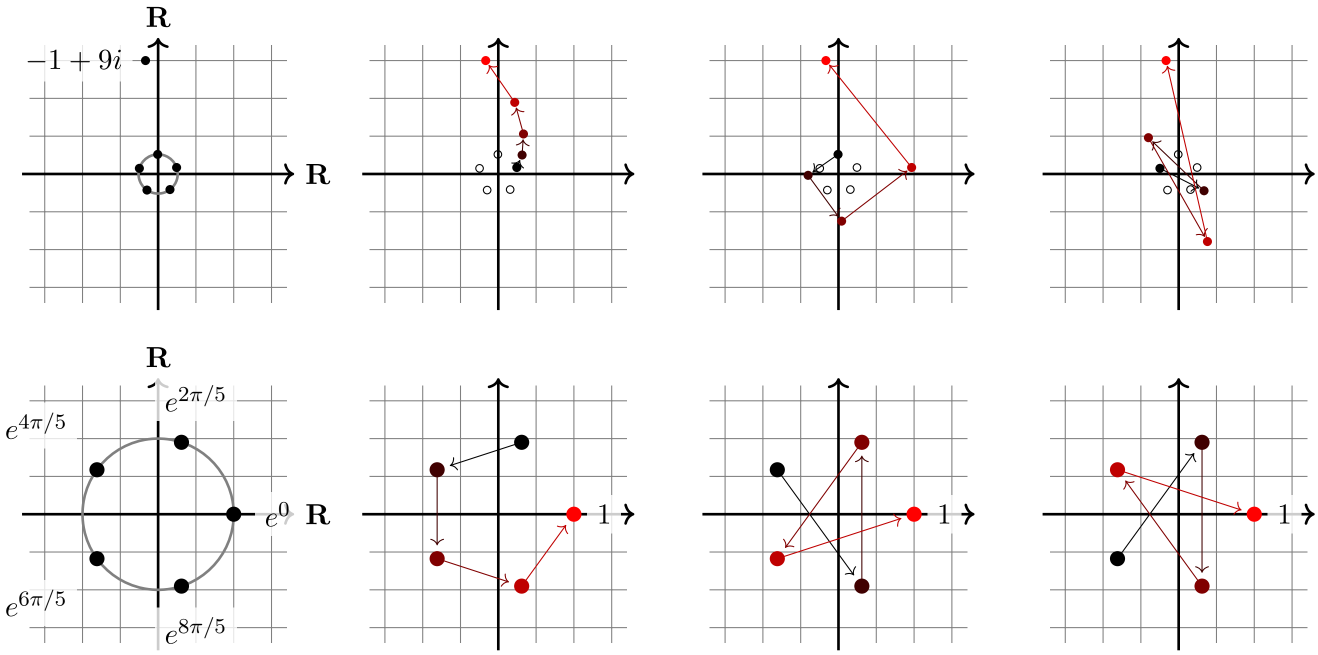
- (De Moivre's theorem) $z^n = (re^{i\theta})^n = r^n e^{in\theta}$
- (complex roots) the nth roots of z are $r^{1/n}e^{i(\theta+2k\pi)/n}$, for every $k = 1, \ldots, n-1$.

For the second point, when $z = 1 + 0i$, then the kth root of z is called the *kth root of unity*.

Inquiry 24.9: This inquiry is about the different forms of complex numbers.

- Express $z = 5\cos(\pi/4) + 5i\sin(\pi/4)$ in standard form.
- Express $w = -\sqrt{3} - i$ in polar form.
- Find the 4th roots of $p = 1 + i$ in Cartesian coordinates.
- Convince yourself that finding nth roots of unity is much easier in polar coordinates than in rectangular coordinates.

Example 24.10. Below are given the 5th roots of $z = -1 + 9i$ and the 5th roots of $z = e^0 = 1$, or unity. For some 5th roots ω of z, the complex numbers $\omega, \omega^2, \omega^3, \omega^4, \omega^5 = z$ are also shown. The circle with radius $\sqrt[5]{|z|}$ is given to emphasize that all 5th roots are the same distance from 0.



Remark 24.11. The space of complex numbers is a 2-dimensional vector space over **R** via the identification of Cartesian coordinates. However, it is a 1-dimensional vector space over **C**.

24.2 Complex matrices

Definition 24.12: Let $\mathbf{z} = [z_1 \ \cdots \ z_n]^T \in \mathbf{C}^n$ be a vector. The (*complex*) *conjugate* is the vector $\overline{\mathbf{z}} = [\overline{z_1} \ \cdots \ \overline{z_n}]^T$.

Often we talk about not just the conjugate, but the *conjugate transpose*. The reason for taking both the conjugate of each element and the transpose, when $n = 2$ and $\mathbf{z} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = x + iy = z$, is to get that

$$\overline{\mathbf{z}}^T\mathbf{z} = \mathbf{z}^*\mathbf{z} = \|\mathbf{z}\|^2 = |z|^2 = \overline{z}z,$$

so the previous notion of length of a vector corresponds with the new notion of absolute value of a complex number. The notation $\mathbf{z}^* = \overline{\mathbf{z}}^T$ is also used for matrices, so that $(A^*)_{ij} = \overline{A_{ji}}$.

Definition 24.13: Let $A \in \mathcal{M}_{n\times n}(\mathbf{C})$. Then

- A is *Hermitian* if $A = A^*$
- A is *unitary* if the columns of A are orthonormal

Proposition 24.14. Let $A \in \mathcal{M}_{n\times n}(\mathbf{C})$ and $\mathbf{z} \in \mathbf{C}^n$. If A is Hermitian, then:

- $\mathbf{z}^*A\mathbf{z}$ is a real number
- every eigenvalue of A is a real number
- eigenvectors (of different eigenvalues are orthogonal

If A is unitary, then:

- $A^*A = I$ and $A^{-1} = A^*$
- every eigenvalue of A is ± 1

Example 24.15. Consider the 2×2 matrix $A = \left[\begin{smallmatrix} 2 & 3-3i \\ 3+3i & 5 \end{smallmatrix}\right]$. This matrix is Hermitian, so should have real eigenvalues and orthogonal eigenvectors by the previous Proposition. Indeed, we find that

$$\det(A - \lambda I) = (2-\lambda)(5-\lambda) - (3-3i)(3+3i) = 10 - 7\lambda + \lambda^2 - 18 = \lambda^2 - 7\lambda - 8 = (\lambda - 8)(\lambda + 1),$$

so the eigenvalues are $\lambda = 8, -1$. For the eigenvectors, we must solve

$$\begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 8z \\ 8w \end{bmatrix} \qquad \Longleftrightarrow \qquad \begin{aligned} -6z + (3-3i)w &= 0, \\ (3+3i)z - 3w &= 0. \end{aligned}$$

Using the first equation to isolate w, we get

$$w = \frac{6z}{3-3i} = \frac{6z}{3-3i}\frac{3+3i}{3+3i} = \frac{(18+18i)z}{9+9} = (1+i)z,$$

which, when placed into the second equation, gives us $(3+3i)z - 3(1+i)z = 0$, which means there are no constraints on z. So we let $z = 1$ and $w = 1+i$. Similarly for the second eigenvector we find $z = 2$ and $w = -1-i$. To check they are orthogonal, we observe that

$$\begin{bmatrix} 1+i \\ 1 \end{bmatrix}^* \cdot \begin{bmatrix} -1-i \\ 2 \end{bmatrix} = \begin{bmatrix} 1-i & 1 \end{bmatrix} \begin{bmatrix} -1-i \\ 2 \end{bmatrix} = (1-i)(-1-i) + 2 = -1 - i + i + i^2 + 2 = -2 + 2 = 0,$$

and we have orthogonality, as desired.

Inquiry 24.16: Consider the *Fourier matrix* $F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}$.

- Note that F is symmetrix. Is it Hermitian? Which entries must change so that it is Hermitian?
- Show that F is unitary.
- Compute the third power F^3 of F.

This matrix will feature heavily in the next lecture.

24.3 Exercises

Exercise 24.1. Show that every complex number $z = x + iy$ for which at least one of x and y are not zero has an inverse. That is, find $w \in \mathbf{C}$ for which $zw = 1$.

Exercise 24.2. Prove all the claims of Proposition 20.4, for $z = x + yi, w = a + bi \in \mathbf{C}$:

1. $\overline{z+w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z}\,\overline{w}$
3. $\overline{\overline{z}} = z$
4. $z + \overline{z} = 2x$
5. $z - \overline{z} = 2yi$
6. $z^{-1} = \overline{z}/|z|^2$ for $z \neq 0$
7. $|z| = 0$ iff $z = 0$
8. $|\overline{z}| = |z|$
9. $|zw| = |z||w|$
10. $|z + w| \leqslant |z| + |w|$

Exercise 24.3. Prove Euler's formula $\cos(\theta) + i\sin(\theta) = e^{i\theta}$ is true by showing that the derivative of $(\cos(\theta) + i\sin(\theta))e^{-i\theta}$ is zero.

Lecture 25: The Fourier series and the discrete Fourier transform

Chapters 8.3 and 9.3 in Strang's "Linear Algebra" and IV.1 in Strang's "Learning from Data"

- Fact 1: The *Fourier series* describes any *function* as a sum of sines and cosines (with real coefficients)
- Fact 2: The *discrete Fourier transform* describes any *sample* (of a function) as a sum of sines and cosines (with complex coefficients)
- Fact 3: Fourer series uses integration, DFT uses matrix multiplication. Both can be simplified for *approximation*.

- Skill 1: Compute the Fourier coefficients of a piecewise continuous function on $[0, 2\pi]$
- Skill 2: Construct the discrete Fourier transform of evenely-spaced data points.

This lecture is all about things named after Joseph Fourier (1768-1830). The key idea of this lecture is how to *approximate* complicated functions in a very simple way. We begin with the more complicated approach, which we resolve with a simplex approach in the second part of the lecture. All functions are define on a finite interval, assumed to be $[0, 2\pi]$.

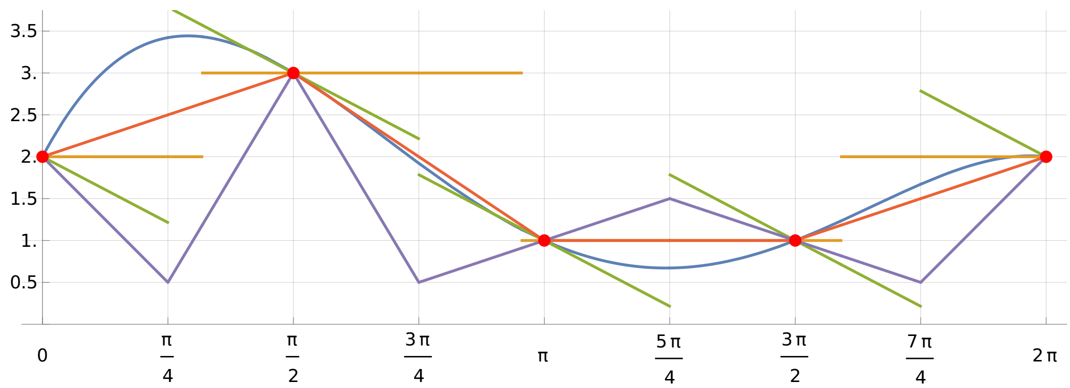
25.1 The Fourier basis and the Fourier series

A function $f : \mathbf{R} \to \mathbf{R}$ is *piecewise continuous* if f is continuous at all except finitely many points of $\mathbf{R}$. We consider the space $PC[0, 2\pi]$ of piecewise continuous functions defined on $[0, 2\pi]$, and make it an inner product space with the inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)\ dx. \tag{15}$$

Piecewise continuous functions can be integrated just like continuous functions, by applying linearity and splitting them up over intervals where they are continuous.

Example 25.1. Consider the following continuous and piecewise continuous functions, which are all different, but coincide at some points.



Of these five functions, three are continuous, but only one is differentiable. One function ise discontinuous at three points, one function is discontinuous at four points. All functions are elements of $PC[0, 2\pi]$. We assume that they are *periodic*, that is, they repeat every interval of length 2π.

Definition 25.2: The *Fourier basis* is the set of functions

$$F = \{1\} \cup \{\sin(nx) \ : \ n \in \mathbf{N}\} \cup \{\cos(nx) \ : \ n \in \mathbf{N}\}.$$

These are all continuous functions in $[0, 2\pi]$. We have yet to show that it is a basis.

Proposition 25.3. The set F is orthogonal.

Inquiry 25.4: This inquiry goes into the the details of Proposition 25.3.

- Compute the inner product, from equation (15), of 1 with $\sin(nx)$ and $\cos(nx)$, to confirm that 1 is orthogonal to all other functions in F.
- Recall the sum of angles formula:

$$\cos(\theta \pm \varphi) = \cos(\theta)\cos(\varphi) \mp \sin(\theta)\sin(\varphi).$$

  Use this to express $\cos(\theta)\cos(\varphi)$ as a sum of only cos functions, and $\sin(\theta)\sin(\varphi)$ only as a difference of cos functions.
- Compute the inner products $\langle \sin(nx), \cos(mx)\rangle$ and $\langle \cos(nx), \cos(mx)\rangle$ for $n \neq m$. You may have to use substitution.

Showing that $\langle \sin(nx), \cos(mx)\rangle = \langle \cos(nx), \cos(mx)\rangle = 0$ for $n \neq m$ shows orthogonality among many (but not all) of the functions in F.

To finish justifying that F is a basis for $PC[0, 2\pi]$, we need to show that F spans this set. Such a proof is beyond the scope of this course, so we continue with the assumption that F is a basis for $PC[0, 2\pi]$.

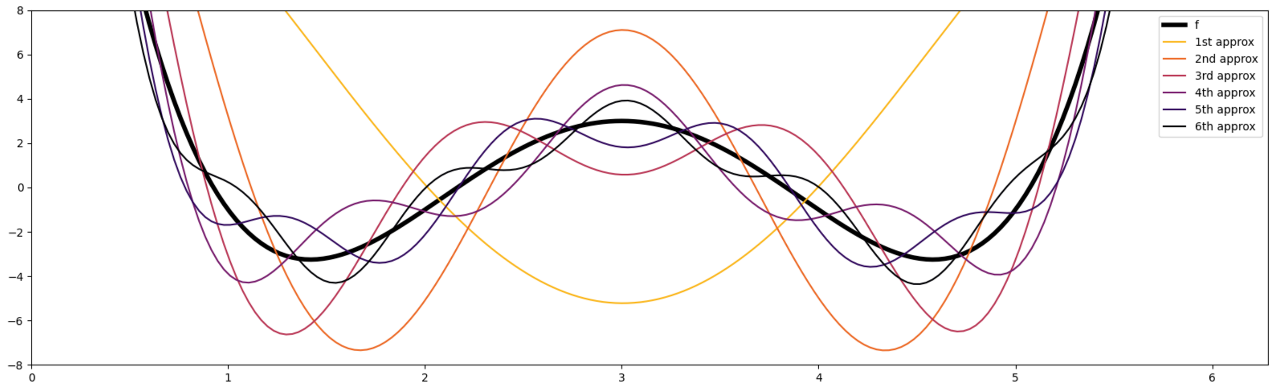
Definition 25.5: Let $f \in PC[0, 2\pi]$. Expressing f using the basis F is the *Fourier series* of f:

$$\begin{aligned} f(x) &= a_0 + a_1 \sin(x) + b_1 \cos(x) + a_2 \sin(2x) + b_2 \cos(2x) + \cdots \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \sin(nx) + b_n \cos(nx)). \end{aligned}$$

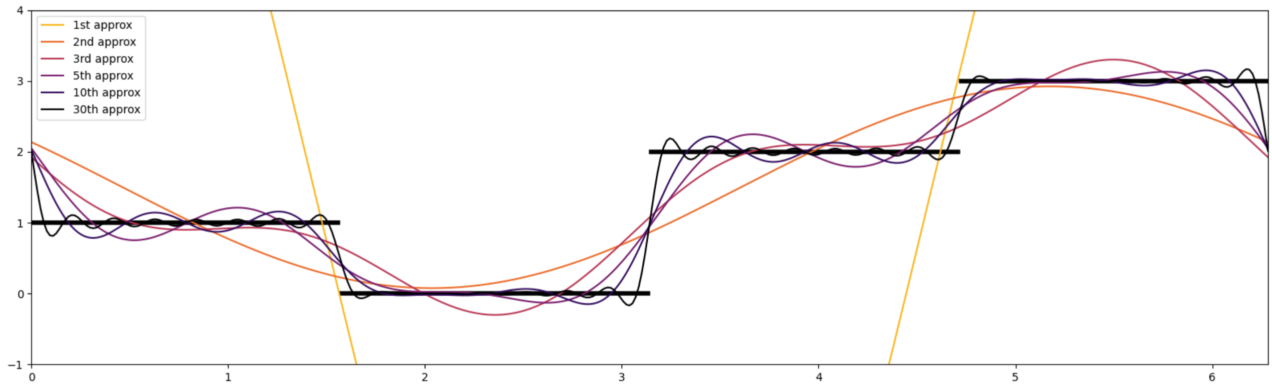
The numbers a_n and b_n are the projections of f onto the vectors spanned by $\sin(nx)$ and $\cos(nx)$, respectively. They are the *Fourier coefficients* of f:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\ dx, \qquad a_n = \frac{\langle f(x), \sin(nx)\rangle}{\|\sin(nx)\|^2}, \qquad b_n = \frac{\langle f(x), \cos(nx)\rangle}{\|\cos(nx)\|^2}.$$

Example 25.6. Unless f is very nice, the sum is usually infinite. Hence we often give only the first few terms in the series to describe f. Here are the first 6 pairs of Fourier coefficients for a simple degree 4 polynomial.



This may seem like overkill, but it is very useful when the orginal function is not continuous everywhere. The Fourier series of any piecewise continuous function will be continuous (and differentiable!) everywhere.



Inquiry 25.7: This inquiry considers a different for of the functions in the Fourier basis. You will need the identities

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{ie^{-i\theta} - ie^{i\theta}}{2}, \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Let $F_e = \{e^{inx} \ : \ n \in \mathbf{Z}\}$.

- Explain why $\text{span}(F) = \text{span}(F_e)$.
- Explain why $\langle e^{inx}, e^{imx}\rangle = 0$ whenever $n \neq m$. This proves orthogonality of F_e.
- Express the $n = 2$ approximation $f(x) = a_0 + a_1 \sin(x) + b_1 \cos(x) + a_2 \sin(2x) + b_2 \cos(2x)$ of f in terms of the elements in F_e.

25.2 The Fourier matrix and the discrete Fourier transform

In Example 25.6 above, we had two functions that were completely known. In the real world, we do not know completely the function we are considering, but only know its value at certain inputs x. A very pertinent queston is then how to convert this discrete data into a continuous function.

Example 25.8. Suppose we have the following data points on the interval $[0, 2\pi]$, evenly spaced out. This could be only part of a signal that we can pick up, or a very sparsely sampled sound:

$$(0,1), \qquad \left(\frac{\pi}{2}, 2\right), \qquad (\pi, -2), \qquad \left(\frac{3\pi}{2}, -3\right).$$

How can we make this data into a continuous function? We could apply the approach from Example 25.6, but we would be assuming the values of the signal at unknown points, and there are several natural ways to extend the discrete signal into a continuous signal.

Definition 25.9: The $n \times n$ *Fourier matrix* $F_n \in \mathcal{M}_{n\times n}(\mathbf{C})$ has $n(F_n)_{ij} = \omega^{(i-1)(j-1)}$, where $\omega = e^{-2\pi i/n}$ is an nth root of unity:

$$F_n = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{n-2} & \cdots & \omega \end{bmatrix}.$$

Given $\mathbf{x} \in \mathbf{R}^n$, the vector $F_n\mathbf{x} \in \mathbf{C}^n$ is called the *discrete Fourier transform* of $\mathbf{x}$.

Remark 25.10. This matrix may look familiar - it is the *Vandermonde* matrix from Definition 11.10 in Lecture 12, for $x_1, \ldots, x_n$ the nth roots of unity. In that lecture the Vandermonde matrix was used to create a polynomial that approximates well some given data points, and here we create a periodic function that approximates well some data points.

Inquiry 25.11: This inquiry is about the 4×4 Fourier matrix $F_4 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{bmatrix}$.

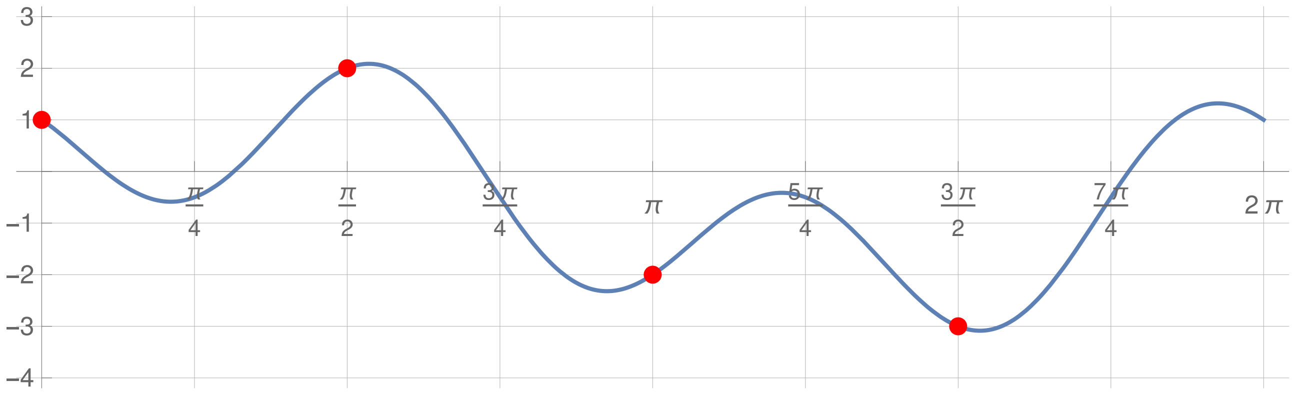
- Show that the columns of F_4 are orthogonal to each other.
- Show that the columns of F_4 all have length 1, by computing the product $\overline{\mathbf{c}_k}^T\mathbf{c}_k = \mathbf{c}_k^*\mathbf{c}_k$ for each column $\mathbf{c}_k$, $k = 1, 2, 3, 4$ of F_4.
- What do the two points above mean for the product $F_4^*F_4$?
- What is the inverse of F_4? What will be the (i, j) entry in the inverse of F_n, for any $n \in \mathbf{N}$?

Here A^* is the conjugate transpose of A.

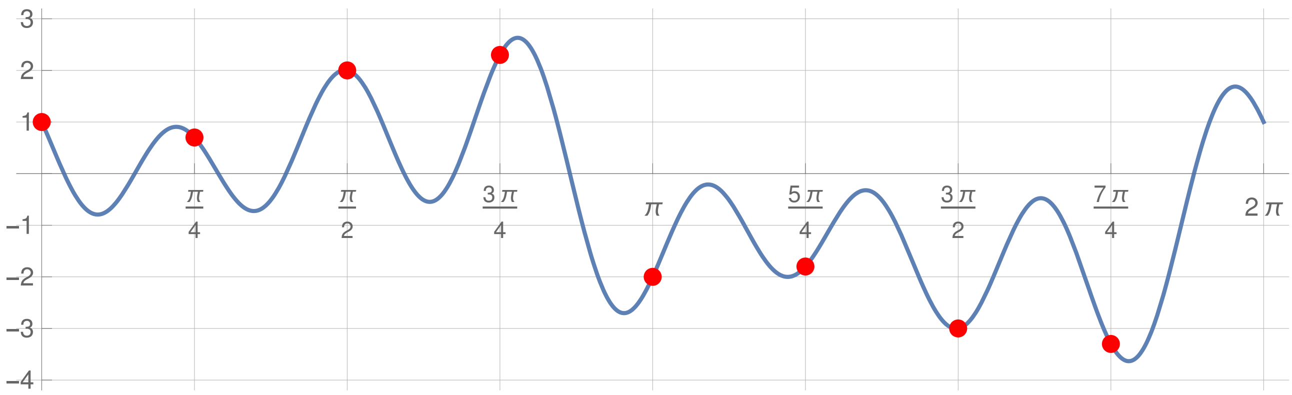
Example 25.12. The reason the Fourier matrix is useful is because it provides the coefficients of the periodic function in the basis F_e that goes through the given data points. So instead of integrating, we simply multiply to get the same result. Consider the data from Example 25.6:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -3 \end{bmatrix}, \qquad F_4\mathbf{x} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -2 \\ -3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 \\ 3-5i \\ 0 \\ 3+5i \end{bmatrix},$$

which means that $f_4(x) = -\frac{1}{2} + \frac{3-5i}{4}e^{ix} + \frac{3+5i}{4}e^{i3x}$. Plotting the real part of this function (since it is complex-valued) along with the original data points gives the following graph:



If we recieved more data points, spaced $\pi/4$ (instead of $\pi/2$) apart, to get a new data vector $\mathbf{x} \in \mathbf{R}^8$, we could use the Fourier matrix F_8 to reconstruct a continuous function from this data:



Inquiry 25.13: This question has you use Python to compute and visualize function approximations. Let $f(x) = \frac{x}{\pi} - 1$.

- Draw this function on the interval $[0, 2\pi]$.
- Compute the Fourier series of f up to $n = 4$.
- Compute the Fourier transform of f using 8 samples.
- Draw the two functions from the privous points using `matplotlib`.

We finish off this lecture with some observations about the Fourier matrix F_n.

Remark 25.14. The Fourier matrix F_n is symmetric, which follows immediately from the definition that $(F_n)_{ij} = \omega^{(i-1)(j-1)}$. The matrix is not Hermitian, as both symmetric and Hermitian would imply that everything off the diagonal is zero. As given, F_n is not unitary, but the columns are orthogonal. For example:

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}^* \begin{bmatrix}1\\i\\-1\\-i\end{bmatrix} = 1 + i - 1 - i = 0, \qquad \begin{bmatrix}1\\i\\-1\\-i\end{bmatrix}^* \begin{bmatrix}1\\-i\\-1\\i\end{bmatrix} = 1 - 1 + 1 - 1 = 0.$$

If we change the coefficient in front from $\frac{1}{n}$ to $\frac{1}{\sqrt{n}}$, then F_n becomes unitary. As a result of the columns being orthogonal, the columns may be interpreted as eigenvectors. Setting all eigenvalues to be 1, we can construct the matrix that has these eigenvectors, and it turns out to be a permutation matrix P

that cycles all the coordinates:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \xrightarrow{P} \begin{bmatrix} y \\ z \\ w \\ x \end{bmatrix} \xrightarrow{P} \begin{bmatrix} z \\ w \\ x \\ y \end{bmatrix} \xrightarrow{P} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \xrightarrow{P} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

25.3 Exercises

Exercise 25.1. Find the Fourier coefficients a_n, b_n up until $n = 3$ for $f(x) = \sin(x)\cos^2(x)$.

Exercise 25.2. Consider the function $f \in C[0, 2\pi]$ given by $f(x) = \left\{ \begin{smallmatrix} -1 & x<\pi \\ 1 & x\geqslant\pi \end{smallmatrix} \right.$.

1. Compute the Fourier series of f up to $n = 1$, $n = 3$, and $n = 5$. Plot these three functions together with f.

2. Compute the discrete Fourier transform of f for $n = 4$, using evenly spaced samples $f(x_k)$ for $x_k = 2k\pi/4$, with $k = 0, 1, 2, 3$. Express it as a sum of sin and cos functions using Euler's formula.

3. Plot the real part of the discrete Fourier transform of f for $n = 4, 8, 12$ together with f. As above, take $4, 8, 12$ evenly spaced samples in the interval $[0, 2\pi]$, starting with 0. You do not need to show your computations.

Part V
Answers to selected lecture exercises

Exercise 1.1. We solve the equation line by line. From the first line, we have $-3b = -5$, which means $b = 5/3$. From the second line on the left and, using the result $a = 35/18$ with the third line on the right, we have:

$$\begin{aligned} 6a - 4b &= 5 \\ 6a - 20/3 &= 5 \\ 18a - 20 &= 15 \\ 18a &= 35 \\ a &= 35/18 \end{aligned} \qquad\qquad \begin{aligned} -a - 5b + c &= -4 \\ -35/18 - 25/3 + c &= -4 \\ -35 - 150 + 18c &= -72 \\ 18c &= 113 \\ c &= 113/18 \end{aligned}$$

Exercise 1.2. By expressing each vector in terms of its constitutent parts, we see the desired result. Let $\mathbf{u} = (u_1, \ldots, u_n)$, $\mathbf{v} = (v_1, \ldots, v_n)$, and $\mathbf{w} = (w_1, \ldots, w_n)$. Then

$$\begin{aligned}
\mathbf{v} \bullet (\mathbf{u} + \mathbf{w}) &= \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \bullet \left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \right) && \text{(definition of vectors } \mathbf{u}, \mathbf{v}, \mathbf{w}\text{)} \\
&= \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \bullet \begin{bmatrix} u_1 + w_1 \\ \vdots \\ u_n + w_n \end{bmatrix} && \text{(definition of matrix addition)} \\
&= v_1(u_1 + w_1) + \cdots + v_n(u_n + w_n) && \text{(definition of dot product)} \\
&= v_1u_1 + v_1w_1 + \cdots + v_nu_n + v_nw_n && \text{(multiplication of real numbers)} \\
&= (v_1u_1 + \cdots + v_nu_n) + (v_1w_1 + \cdots + v_nw_n) && \text{(rearranging)} \\
&= \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \bullet \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \bullet \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} && \text{(definition of dot product)} \\
&= \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{w}.
\end{aligned}$$

Exercise 1.6. Since $\mathbf{w} = (w_1, w_2, w_3)$ and $\mathbf{z} = (z_1, z_2, z_3)$ are perpendicular to $\mathbf{v} = (v_1, v_2, v_3)$, we have that

$$\begin{aligned} 0 &= \mathbf{v} \bullet \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3, \\ 0 &= \mathbf{v} \bullet \mathbf{z} = v_1z_2 + v_2z_2 + v_3z_3. \end{aligned}$$

The halfway point between $\mathbf{w}$ and $\mathbf{z}$ is $\mathbf{h} = \left(\frac{w_1+z_1}{2}, \frac{w_2+z_2}{2}, \frac{w_3+z_3}{2}\right)$, and for this vector

$$\begin{aligned}
\mathbf{v} \bullet \mathbf{h} &= (v_1, v_2, v_3) \bullet \left(\frac{w_1+z_1}{2}, \frac{w_2+z_2}{2}, \frac{w_3+z_3}{2}\right) \\
&= v_1 \bullet \frac{w_1+z_1}{2} + v_2 \bullet \frac{w_2+z_2}{2} + v_3 \bullet \frac{w_3+z_3}{2} \\
&= \frac{1}{2}\left(v_1w_1 + v_1z_1 + v_2w_2 + v_2z_2 + v_3w_3 + v_3z_3\right) \\
&= \frac{1}{2}\left((v_1w_1 + +v_2w_2 + v_3w_3) + (v_1z_1 + v_2z_2 + v_3z_3)\right) \\
&= \frac{1}{2}\left(\mathbf{v} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{z}\right) \\
&= \frac{1}{2}(0+0) \\
&= 0.
\end{aligned}$$

Exercise 2.4. Let $A, B \in \mathcal{M}_{n\times n}$, with ij-entries a_{ij} and b_{ij}, respectively.

1. Suppose that A, B are lower triangular, so $a_{ij} = 0$ and $b_{ij} = 0$ if $i < j$. In the product, the ij entry of AB, for $i < j$, is

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \left(\sum_{k=1}^{i} A_{ik} \underbrace{B_{kj}}_{=0}\right) + \left(\sum_{k=i+1}^{n} \underbrace{A_{ik}}_{=0} B_{kj}\right) = 0.$$

   Hence AB is also lower triangular.

2. Suppose that A, B are upper triangular, so $a_{ij} = 0$ and $b_{ij} = 0$ if $i > j$. In the product, the ij entry of AB, for $i > j$, is

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \left(\sum_{k=1}^{j} \underbrace{A_{ik}}_{=0} B_{kj}\right) + \left(\sum_{k=j+1}^{n} A_{ik} \underbrace{B_{kj}}_{=0}\right) = 0.$$

   Hence AB is also upper triangular.

3. The result does not have to be triangular, for example:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

   Here we have two non diagonal matrices, whose product is a diagonal matrix:

$$\begin{bmatrix} 6 & -10 \\ 77 & 22 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 82 & 0 \\ 0 & 451 \end{bmatrix}.$$

Exercise 3.1. 1. We describe each step of Gaussian elimination in terms of elementary matrices:

clear the second row below the first pivot:
$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 & 5 \\ 6 & -2 & -1 & -2 \\ 1 & -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & -5 & -12 \\ 1 & -1 & -1 & -3 \end{bmatrix}$$

clear the third row below the first pivot:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & -5 & -12 \\ 1 & -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & -5 & -12 \\ 0 & -\frac{2}{3} & -\frac{5}{3} & -\frac{14}{3} \end{bmatrix}$$

swap the second and third rows:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & -5 & -12 \\ 0 & -\frac{2}{3} & -\frac{5}{3} & -\frac{14}{3} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & -\frac{2}{3} & -\frac{5}{3} & -\frac{14}{3} \\ 0 & 0 & -5 & -12 \end{bmatrix}$$

Since all the pivots have zeros below them, we have finished Gaussian elimintation.

2. We describe each step of Gauss–Jordan elimination in terms of elementary matrices:

clear the first row above the second pivot:
$$\begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & -\frac{2}{3} & -\frac{5}{3} & -\frac{14}{3} \\ 0 & 0 & -5 & -12 \end{bmatrix} = \begin{bmatrix} 3 & 0 & \frac{9}{2} & 12 \\ 0 & -\frac{2}{3} & -\frac{5}{3} & -\frac{14}{3} \\ 0 & 0 & -5 & -12 \end{bmatrix}$$

clear the second row above the third pivot:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & \frac{9}{2} & 12 \\ 0 & -\frac{2}{3} & -\frac{5}{3} & -\frac{14}{3} \\ 0 & 0 & -5 & -12 \end{bmatrix} = \begin{bmatrix} 3 & 0 & \frac{9}{2} & 12 \\ 0 & -\frac{2}{3} & 0 & -\frac{2}{3} \\ 0 & 0 & -5 & -12 \end{bmatrix}$$

clear the first row above the third pivot:
$$\begin{bmatrix} 1 & 0 & \frac{9}{10} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & \frac{9}{2} & 12 \\ 0 & -\frac{2}{3} & 0 & \frac{2}{3} \\ 0 & 0 & -5 & -12 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & \frac{6}{5} \\ 0 & -\frac{2}{3} & 0 & -\frac{2}{3} \\ 0 & 0 & -5 & -12 \end{bmatrix}$$

Since all pivots havce zeros below them, we have finished Gauss–Jordan elimination. To make the solution a bit easier to find, we already here multiply by diagonal matrices, to make the diagonal entries be 1:

make the $(1,1)$-entry be 1:
$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & \frac{6}{5} \\ 0 & -\frac{2}{3} & 0 & -\frac{2}{3} \\ 0 & 0 & -5 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{2}{5} \\ 0 & -\frac{2}{3} & 0 & -\frac{2}{3} \\ 0 & 0 & -5 & -12 \end{bmatrix}$$

make the $(2,2)$-entry be 1:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \frac{2}{5} \\ 0 & -\frac{2}{3} & 0 & -\frac{2}{3} \\ 0 & 0 & -5 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{2}{5} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -5 & -12 \end{bmatrix}$$

make the $(3,3)$-entry be 1:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \frac{2}{5} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -5 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{2}{5} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{12}{5} \end{bmatrix}$$

We immediately find the solution to be $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2/5 \\ 1 \\ 12/5 \end{bmatrix}$.

3. This step is just (careful) matrix multiplication:

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&-\frac{1}{5}\end{bmatrix}\begin{bmatrix}1&0&0\\0&-\frac{3}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}\frac{1}{3}&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{\text{step 3: make the diagonal entries be 1}}\underbrace{\begin{bmatrix}1&0&\frac{9}{10}\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&-\frac{1}{3}\\0&0&1\end{bmatrix}\begin{bmatrix}1&-\frac{3}{2}&0\\0&1&0\\0&0&1\end{bmatrix}}_{\text{step 2: Gauss–Jordan elimination}}\underbrace{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-\frac{1}{3}&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}_{\text{step 1: Gaussian elimination}}$$

$$= \underbrace{\begin{bmatrix}\frac{1}{3}&0&0\\0&-\frac{3}{2}&0\\0&0&-\frac{1}{5}\end{bmatrix}}_{\text{step 3}}\underbrace{\begin{bmatrix}1&-\frac{3}{2}&\frac{9}{10}\\0&1&-\frac{1}{3}\\0&0&1\end{bmatrix}}_{\text{step 2}}\underbrace{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\-\frac{1}{3}&0&1\end{bmatrix}}_{\text{step 1}}$$

$$= \begin{bmatrix}\frac{1}{3}&-\frac{1}{2}&\frac{3}{10}\\0&-\frac{3}{2}&\frac{1}{2}\\0&0&-\frac{1}{5}\end{bmatrix}\begin{bmatrix}1&0&0\\-\frac{1}{3}&0&1\\-2&1&0\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{1}{10}&\frac{3}{10}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}&-\frac{3}{2}\\\frac{2}{5}&-\frac{1}{5}&0\end{bmatrix}.$$

This is the inverse of A.

Exercise 3.2. Some examples are given below. Many more exist.

1. An example is $\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$, as all the pivots can be read off the diagonal.

2. An example is $\begin{bmatrix}1&0&0\\6&2&0\\5&8&3\end{bmatrix}$, as elimination tells us to:

   subtract $\ell_{21} = 6$ of the first row from the second row: $\begin{bmatrix}1&0&0\\0&2&0\\5&8&3\end{bmatrix}$

   subtract $\ell_{31} = 5$ of the first row from the third row: $\begin{bmatrix}1&0&0\\0&2&0\\0&8&3\end{bmatrix}$

   subtract $\ell_{32} = 4$ of the second row from the third row: $\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$

   This ends Gaussian elimination, as the matrix is upper triangular, and indicates the pivots on the diagonal.

3. An example is $\begin{bmatrix}1&1&1\\1&3&1\\1&1&1\end{bmatrix}$, as elimination tells us to:

   subtract $\ell_{21} = 1$ of the first row from the second row: $\begin{bmatrix}1&1&1\\0&2&0\\1&1&1\end{bmatrix}$

   subtract $\ell_{31} = 1$ of the first row from the second row: $\begin{bmatrix}1&1&1\\0&2&0\\0&0&0\end{bmatrix}$

   This ends Gaussian elimination, as the matrix is upper triangular, and indicates the pivots on the diagonal. There is no third pivot, since the third row is all zeros.

Exercise 3.3. 1. For the first matrix, elimination tells us to:

subtract $\ell_{21} = d/a$ of the first row from the second row: $\begin{bmatrix} a & b & c \\ 0 & e - bd/a & f - cd/a \\ g & h & i \end{bmatrix}$

subtract $\ell_{31} = g/a$ of the first row from the third row: $\begin{bmatrix} a & b & c \\ 0 & e - bd/a & f - cd/a \\ 0 & h - bg/a & i - cg/a \end{bmatrix}$

For the multiplier ℓ_{32}, it needs to be

$$(h - bg/a) \cdot (e - bd/a)^{-1} = \frac{h - \frac{bg}{a}}{e - \frac{bd}{a}} = \frac{ah - bg}{ae - bd}.$$

The lower right entry after this step will be $(i - cg/a) - (f - cd/a) \cdot \frac{ah-bg}{ae-bd}$, which we call simply n, because it is very long to write. So elimination tells us to

subtract $\ell_{32} = \dfrac{ah - bg}{ae - bd}$ of the second row from the third row: $\begin{bmatrix} a & b & c \\ 0 & e - bd/a & f - cd/a \\ 0 & 0 & n \end{bmatrix}$

This ends Gaussian elimination, as the matrix is upper triangular, and indicates the pivots $a, e - bd/a, n$ on the diagonal.

For the second matrix, elimination tells us to:

subtract $\ell_{32} = h/e$ of the second row from the third row: $\begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 0 & 0 & i - fh/e \end{bmatrix}$

This ends Gaussian elimination, as the matrix is upper triangular, and indicates the pivots $e, i - fh/e$ on the diagonal.

For the third matrix, elimination tells us to:

subtract $\ell_{21} = d/a$ of the first row from the second row: $\begin{bmatrix} a & b & c \\ 0 & 0 & f - cd/a \\ d & bd/a & i \end{bmatrix}$

subtract $\ell_{31} = d/a$ of the first row from the third row: $\begin{bmatrix} a & b & c \\ 0 & 0 & f - cd/a \\ 0 & 0 & i - cd/a \end{bmatrix}$

This ends Gaussian elimination, as the matrix is upper triangular, and indicates the pivots $a, i - cd/a$ on the diagonal.

For the fourth matrix, elimination tells us to:

subtract $\ell_{32} = 1$ of the second row from the third row: $\begin{bmatrix} 0 & b & c \\ 0 & e & ce/b \\ 0 & 0 & 0 \end{bmatrix}$

This ends Gaussian elimination, as the matrix is upper triangular, and indicates the only pivot e on the diagonal.

2. Here is an example of such a function, in Python, using the input `A[[a,b,c],[d,e,f],[g,h,i]]`. We use the result from the first matrix in part 1. above.

```
def pivots(A):
    a = A[0][0]
```

```
    b = A[0][1]
    c = A[0][2]
    d = A[1][0]
    e = A[1][1]
    f = A[1][2]
    g = A[2][0]
    h = A[2][1]
    i = A[2][2]
    return [a, b*d/a, (i-c*g/a)-(f-c*d/a)*(a*h-b*g)(a*e-b*d)]
```

3. Here is some Python code that produces the range and average as requested, using the function above.

```
import numpy as np
values = []
for i in range(1000):
    M1 = np.random.rand(3,3)
    M2 = 2*M1 - np.ones((3,3))
    values += pivots(M2)
print([min(values), max(values), sum(values)/len(values)])
```

This is the result it prints on one particular run:

```
[-1105.1138842178975, 1650.5842938466174, -0.272518610029052]
```

Exercise 3.6. We apply row operations to the block matrix $[A\ I] = \begin{bmatrix} 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & 0 & -4 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{bmatrix}$, as below.

swap the first and the second rows to get a first pivot: $\begin{bmatrix} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{bmatrix}$

subtract $\ell_{31} = 2$ of the first row from the third row: $\begin{bmatrix} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 10 & 0 & -2 & 1 \end{bmatrix}$

subtract $\ell_{32} = 1$ of the second row from the third row: $\begin{bmatrix} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 11 & -1 & -2 & 1 \end{bmatrix}$

This finishes Gaussian elminaton, so we proceeed with Gauss–Jordan elimination above the diagonal.

subtract $\ell_{23} = -1/11$ of the third row from the second row: $\begin{bmatrix} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & 0 & 10/11 & -2/11 & 1/11 \\ 0 & 0 & 11 & -1 & -2 & 1 \end{bmatrix}$

subtract $\ell_{13} = -4/11$ of the third row from the first row: $\begin{bmatrix} 1 & 0 & 0 & -4/11 & 3/11 & 1/11 \\ 0 & 2 & 0 & 10/11 & -2/11 & 1/11 \\ 0 & 0 & 11 & -1 & -2 & 1 \end{bmatrix}$

multiply each row by the inverse of the pivots: $\begin{bmatrix} 1 & 0 & 0 & -4/11 & 3/11 & 1/11 \\ 0 & 1 & 0 & 5/11 & -1/11 & 1/22 \\ 0 & 0 & 1 & -1/11 & -2/11 & 1/11 \end{bmatrix}$

Hence the inverse of A is $A^{-1} = \begin{bmatrix} -4/11 & 3/11 & 1/11 \\ 5/11 & -1/11 & 1/22 \\ -1/11 & -2/11 & 1/11 \end{bmatrix}$.

Exercise 4.1. The matrix L is the inverse of the row reduction matrices. When it is on the left, we have:

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&\frac{8}{3}&1\end{bmatrix}}_{\text{clear the }(3,2)\text{-entry}}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\\frac{1}{2}&0&1\end{bmatrix}}_{\text{clear the }(3,1)\text{-entry}}\underbrace{\begin{bmatrix}1&0&0\\-\frac{1}{6}&1&0\\0&0&1\end{bmatrix}}_{\text{clear the }(2,1)\text{-entry}}\begin{bmatrix}6&0&-2\\1&3&4\\-3&-8&2\end{bmatrix}=\begin{bmatrix}6&0&-2\\0&2&13/3\\0&0&113/9\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\\frac{1}{2}&\frac{8}{3}&1\end{bmatrix}\begin{bmatrix}1&0&0\\-\frac{1}{6}&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}6&0&-2\\1&3&4\\-3&-8&2\end{bmatrix}=\begin{bmatrix}6&0&-2\\0&2&13/3\\0&0&113/9\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\-\frac{1}{6}&1&0\\\frac{5}{9}&\frac{8}{3}&1\end{bmatrix}\begin{bmatrix}6&0&-2\\1&3&4\\-3&-8&2\end{bmatrix}=\begin{bmatrix}6&0&-2\\0&2&13/3\\0&0&113/9\end{bmatrix}.$$

To find the inverse of this matrix, we can perform Gaussian elimination, exactly as above:

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-\frac{8}{3}&1\end{bmatrix}}_{\text{clear the }(3,2)\text{-entry}}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\-\frac{5}{9}&0&1\end{bmatrix}}_{\text{clear the }(3,1)\text{-entry}}\underbrace{\begin{bmatrix}1&0&0\\\frac{1}{6}&1&0\\0&0&1\end{bmatrix}}_{\text{clear the }(2,1)\text{-entry}}\begin{bmatrix}1&0&0\\-\frac{1}{6}&1&0\\\frac{5}{9}&\frac{8}{3}&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\-\frac{5}{9}&-\frac{8}{3}&1\end{bmatrix}\begin{bmatrix}1&0&0\\\frac{1}{6}&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-\frac{1}{6}&1&0\\\frac{5}{9}&\frac{8}{3}&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\\frac{1}{6}&1&0\\-1&-\frac{8}{3}&1\end{bmatrix}\begin{bmatrix}1&0&0\\-\frac{1}{6}&1&0\\\frac{5}{9}&\frac{8}{3}&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

This is the inverse U. Hence $a=\frac{1}{6}$, $b=-1$, $c=-\frac{5}{9}$.

Exercise 4.2. We immediately see that the first row operation will give two leading zeros in row 2, so we swap rows 2 and 3:

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}3&1&1\\3&1&3\\1&1&3\end{bmatrix}}_{A}=\begin{bmatrix}3&1&1\\1&1&3\\3&1&3\end{bmatrix}.$$

Then we begin Gaussian elimination:

$$\text{clear the second row below the first pivot:}\quad\begin{bmatrix}1&0&0\\-\frac{1}{3}&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}3&1&1\\1&1&3\\3&1&3\end{bmatrix}=\begin{bmatrix}3&1&1\\0&\frac{2}{3}&\frac{8}{3}\\3&1&3\end{bmatrix}$$

$$\text{clear the third row below the first pivot:}\quad\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}3&1&1\\0&\frac{2}{3}&\frac{8}{3}\\3&1&3\end{bmatrix}=\begin{bmatrix}3&1&1\\0&\frac{2}{3}&\frac{8}{3}\\0&0&2\end{bmatrix}.$$

Separating the right side into DU, we get the following equation:

$$\underbrace{\begin{bmatrix}1&0&0\\-\frac{1}{3}&1&0\\-1&0&1\end{bmatrix}}_{L^{-1}}\underbrace{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}3&1&1\\3&1&3\\1&1&3\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}3&0&0\\0&\frac{3}{2}&0\\0&0&2\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}1&\frac{1}{3}&\frac{1}{3}\\0&1&\frac{16}{9}\\0&0&1\end{bmatrix}}_{U}.$$

The inverse of L^{-1} is the same as L^{-1}, just with negative values in the off-diagonals. Hence we get

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}3 & 1 & 1\\ 3 & 1 & 3\\ 1 & 1 & 3\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ \frac{1}{3} & 1 & 0\\ 1 & 0 & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}3 & 0 & 0\\ 0 & \frac{3}{2} & 0\\ 0 & 0 & 2\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}1 & \frac{1}{3} & \frac{1}{3}\\ 0 & 1 & \frac{16}{9}\\ 0 & 0 & 1\end{bmatrix}}_{U}$$

as the answer.

Exercise 4.3. In Example 3.8, we had the following result:

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}}_{E_{32}} \cdot \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \frac{1}{2} & 0 & 1\end{bmatrix}}_{E_{31}} \cdot \underbrace{\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}}_{P_{12}} \cdot \begin{bmatrix}0 & 6 & -2 & 2\\ 4 & 8 & -4 & 8\\ -2 & 2 & 7 & 12\end{bmatrix} = \begin{bmatrix}4 & 8 & -4 & 8\\ 0 & 6 & -2 & 2\\ 0 & 0 & 7 & 14\end{bmatrix}$$

For $PA = LDU$ decomposition, we don't need the fourth column **b** used in this example. We also note several necessary things:

$$D = \begin{bmatrix}4 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 7\end{bmatrix}, \qquad E_{32}^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}, \qquad E_{31}^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -\frac{1}{2} & 0 & 1\end{bmatrix}.$$

For the inverses of elementary matrices, we used the observations from Example **??**. This gets us almost where we want to be:

$$\underbrace{\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}0 & 6 & -2\\ 4 & 8 & -4\\ -2 & 2 & 7\end{bmatrix}}_{A} = E_{31}^{-1}E_{32}^{-1}\begin{bmatrix}4 & 8 & -4\\ 0 & 6 & -2\\ 0 & 0 & 7\end{bmatrix}.$$

The lower triangular matrix is

$$L = E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -\frac{1}{2} & 1 & 1\end{bmatrix},$$

and the product DU is

$$\begin{bmatrix}4 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 7\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & -1/3\\ 0 & 0 & 1\end{bmatrix}.$$

Putting this all together, we get

$$\underbrace{\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}0 & 6 & -2\\ 4 & 8 & -4\\ -2 & 2 & 7\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -\frac{1}{2} & 1 & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}4 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 7\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & -1/3\\ 0 & 0 & 1\end{bmatrix}}_{U}.$$

Exercise 4.4. Each one of the elementary matrices from Definition 3.7 has an inverse:

inverse of permutation is itself:
$$\begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix}^{-1}=\begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$$

inverse of row operations is opposite operation:
$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\-\frac{2}{5}&0&1&0\\0&0&0&1\end{bmatrix}^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\\frac{2}{5}&0&1&0\\0&0&0&1\end{bmatrix}$$

inverse of multiplication is reciprocal multiplication:
$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\frac{1}{10}\end{bmatrix}^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&10\end{bmatrix}$$

Although these are examples, the method clearly generalizes. Since each E_i has an inverse, the product of their inverses is the inverse of A. That is,

$$\begin{aligned}\underbrace{\left(E_k^{-1}\cdots E_2^{-1}\cdot E_1^{-1}\right)}_{A^{-1}}\underbrace{\left(E_1\cdot E_2\cdots E_k\right)}_{A} &= E_k^{-1}\cdots E_2^{-1}\cdot\underbrace{E_1^{-1}\cdot E_1}_{I}\cdot E_2\cdots E_k\\ &= E_k^{-1}\cdots E_2^{-1}\cdot E_3\underbrace{E_2^{-1}\cdot E_2}_{I}\cdot E_3\cdots E_k\\ &\ \vdots\\ &= I.\end{aligned}$$

Exercise 5.1. The operations of addition and scalar multiplication clearly exist:

- $c_1(2,1)+c_2(2,1)=(c_1+c_2)(2,1)$, and $c_1+c_2\in\mathbf{R}$
- $c_1\cdot(c_2(2,1))=(c_1c_2)(2,1)$, and $c_1c_2\in\mathbf{R}$

The identity element is the zero vector $(0,0)=0(2,1)$, and every $c(2,1)$ has an inverse $(-c)(2,1)$, for which $c(2,1)+(-c)(2,1)=(c+(-c))(2,1)=0(2,1)=(0,0)$. Finally, scalar multiplication has the usual identity 1, as $1(c(2,1))=(1\cdot c)(2,1)=c(2,1)$. Commutativity, associativity, and distributivity in this space all follow from the same properties of $\mathbf{R}^2$ as a vector space.

Exercise 5.2. To show this, we need to show that every element in W can be expressed an element in V. An arbitrary element of W looks like

$$a(\mathbf{u}+\mathbf{v})+b(\mathbf{v}+\mathbf{w}),$$

for some $a,b\in\mathbf{R}$. Rearranging, we get

$$a\mathbf{u}+(a+b)\mathbf{v}+b\mathbf{w},$$

which is an element of the span of $\mathbf{u},\mathbf{v},\mathbf{w}$, hence in V. Therefore $W\subseteq V$.

Exercise 5.3. 1. This is not a vector space, because scalar multiplication is not distributive over field addition. For example, if $f(x)=x^2-1$, $a=3$, $b=-2$, then

$$\begin{aligned}(a+b)f(x)&=(3+(-2))f(x)=1f(x)=f(1x)=x^2-1,\\ af(x)+bf(x)&=3f(x)+(-2)f(x)=f(3x)+f(-2x)=(3x)^2-1+(-2x)^2-1\\ &=9x^2+4x^2-2=13x^2-2,\end{aligned}$$

and these are clearly not the same function.

2. This is not a vector space, because addition is not commutative. For example, if $f(x) = x^2$ and $g(x) = 2x$, then

$$\begin{aligned} f + g &= f(g(x)) = f(2x) = 4x^2, \\ g + f &= g(f(x)) = g(x^2) = 2x^2, \end{aligned}$$

which are clearly not the same function.

Exercise 5.4. We take advantage of the fact that there is a 3×3 identity matrix in columns 3-5, and consider them as our pivot columns. That is, columns 1,2,6 are free columns. The first free column gives us the first vector, as

$$\begin{bmatrix} 2 & 9 & 1 & 0 & 0 & 9 \\ 0 & -3 & 0 & 1 & 0 & -3 \\ 8 & -6 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ -8 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The second free column gives us the second vector, as

$$\begin{bmatrix} 2 & 9 & 1 & 0 & 0 & 9 \\ 0 & -3 & 0 & 1 & 0 & -3 \\ 8 & -6 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -9 \\ 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

And the third free column gives us the third vector, as

$$\begin{bmatrix} 2 & 9 & 1 & 0 & 0 & 9 \\ 0 & -3 & 0 & 1 & 0 & -3 \\ 8 & -6 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -9 \\ 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

These are all linearly independent, as witnessed by rows 1,2,6, which are zeros for all but exactly one vector. Hence

$$\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -9 \\ 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -9 \\ 3 \\ -1 \\ 1 \end{bmatrix} \right).$$

Exercise 5.5.

1. Every element in span(V) is a sum

$$\sum_{\mathbf{v} \in V} a_{\mathbf{v}} \mathbf{v},$$

for $a_{\mathbf{v}} \in \mathbf{R}$ (technically, we must have only finitely many $a_{\mathbf{v}}$ be nonzero, but that is not an issue here). Since V is a linear combination of vectors, every element of V appears in span(V). Conversely, given $\mathbf{x} \in V$, the sum equals $\mathbf{x}$ when all other coefficients $a_{\mathbf{y} \neq \mathbf{x}}$ are 0 and $a_{\mathbf{x}} = 1$.

For the zero vector space, linear combinations $a \cdot 0$ for any $a \in \mathbf{R}$ always give back 0, so there is nothing else in the span.

2. The following matrices work (there are others for A, but B is unique):

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

They can not be the same matrix because A must have 3 pivots (no free columns) and B must have 0 pivots (3 free columns).

Exercise 5.6. 1. First we observe the following linear combinations from the two spans:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

These are useful because they only have one nonzero entry. That is,

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + (-y) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \\ &= x \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) + z \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) + (-y) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \\ &= x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (x+z-y) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Hence $\mathbf{R}^3$ is a subspace of $\subseteq V + W$.

2. We take a linear combination of vectors from both spans. Consider

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

This is an element of $V + W$. For it to be an element of $V \cup W$, it must either be in V or in W. This vector is in V if any only if the matrix equation

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

has a solution. However, the first line is the equation $x_1 = 0$ and the last line is $x_2 = -1$, so it must be that $x_1 + x_2 = -1$. But the second line says $x_1 + x_2 = 1$, and these two equations contradict each other, so there is no solution. Similarly, this vector is in W if and only if the matrix equation

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

has a solution. Again, we find the first line $x_1 = 0$ and the third line $x_1 = -1$ cannot both be true at the same time, hence there is no solution. Therefore $\mathbf{u} \notin V$ and $\mathbf{u} \notin W$, so $\mathbf{u} \notin V \cup W$. Since $\mathbf{u} \in V + W$, it follows that $V \cup W \neq V + W$.

Exercise 6.1. The product is

$$\mathbf{v}\mathbf{w}^T = \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{bmatrix}.$$

if $a = 0$, then we have the zero matrix, which has rank 0. But if a is any nonzero real number, then the the reduced row echelon form of A will be

$$\begin{bmatrix} a & a & a & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which clearly has only one pivot. So in this case, the rank is 1.

Exercise 6.2. First we find the particular solutions. We get these by elimination on the augmented matrix $[A\ \mathbf{b}]$. The first multiplier is $\ell_{21} = 2$:

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -9 & -3 & 0 & 9 \\ 6 & 0 & -21 & 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -9 & -3 & 0 & 9 \\ 0 & 0 & -3 & 6 & 2 & -19 \end{bmatrix}.$$

We see the pivots already as $3, -3$. Now we clear the -9 above the -3:

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -9 & -3 & 0 & 9 \\ 0 & 0 & -3 & 6 & 2 & -19 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & -21 & -6 & 66 \\ 0 & 0 & -3 & 6 & 2 & -19 \end{bmatrix}.$$

Finally we multiply by the reciprocals of the pivots:

$$\begin{bmatrix} 1/3 & 0 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & -21 & -6 & 66 \\ 0 & 0 & -3 & 6 & 2 & -19 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -7 & -2 & 22 \\ 0 & 0 & 1 & -2 & -2/3 & 19/3 \end{bmatrix}.$$

We find the particular solution immediately by placing the last column $\mathbf{d}$ in the pivot variable spots, and get $\mathbf{p} = \begin{bmatrix} 22 & 0 & 19/3 & 0 & 0 \end{bmatrix}^T$. The special solutions, which we know there are 3 (as there are 3 free columns), come from considering $R\mathbf{x} = 0$. The three special solutions will have one 1 in each of the free variable spots, and 0 in the other free variable spots.

$$\begin{bmatrix} 1 & 0 & 0 & -7 & -2 \\ 0 & 0 & 1 & -2 & -2/3 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1 = 0,\ x_3 = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & -7 & -2 \\ 0 & 0 & 1 & -2 & -2/3 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1 = 7,\ x_3 = 2$$

$$\begin{bmatrix} 1 & 0 & 0 & -7 & -2 \\ 0 & 0 & 1 & -2 & -2/3 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1 = 2,\ x_3 = 2/3$$

Hence the complete solution is

$$\mathbf{x} = \begin{bmatrix} 22 \\ 0 \\ 19/3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{bmatrix},$$

for any $x_2, x_4, x_5 \in \mathbf{R}$.

Exercise 6.3. Note the answer is presented in the usual (particular solution) + (special solution) way. The free column is the second column of A, since the only value 1 is in the second row of the only special solution.

1. In a particular solution the free variables are zero, which occurs in

$$\begin{bmatrix} 19 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -2 \end{bmatrix} + (-4) \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

This vector is still on the line of intersection, and we can build $A\mathbf{x} = \mathbf{b}$ from it. The augmented matrix $[R\ \mathbf{d}]$ from the equation $R\mathbf{x} = \mathbf{d}$, obtained via elimination, is

$$\begin{bmatrix} 1 & 3 & 0 & 19 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

Here $R = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ could already be A, and $\mathbf{d} = \begin{bmatrix} 19 \\ 0 \\ -2 \end{bmatrix}$ could already be $\mathbf{b}$.

2. We can simply add rows together to get rid of the zeros:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 17 \\ 32 \end{bmatrix}.$$

Exercise 7.1. Choosing 3 vectors from 5 gives $\binom{5}{3} = 10$ choices. Among the five vectors, we see the following linear dependence equations:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

The first equation gives 3 linearly dependent sets of size 3, and the second gives 1. Replacing $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ with $\frac{1}{2} \begin{bmatrix} 2 & 0 & 2 \end{bmatrix}^T$ in the second equation gives another linearly dependent set of size 3. Hence we should find $10 - 3 - 1 - 1 = 5$ linearly independent sets of size 3. The linearly independent sets of size 3 cotaining $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ are:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Further, linearly independent sets of size 3 containing $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ that have not already been given are:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}, \quad \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\}, \quad \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

This gives 5 different linearly independent sets of size 3 (linear independence can be checked by row reduction), so we are done.

Exercise 7.2. Here we again use Python, and take a 2×2 matrix to be a list of lists `[[a,b],[c,d]]`.

1. The following function takes a 2×2 matrix as input and returns `True` if one column is a multiple of the other, and `False` otherwise. We have an additional function that allows for computer precision up to 10 decimal points.

   ```
   def iszero(n):
       return (abs(n) < 1e-10)


   def twomult(mat):
       ratio1 = mat[0][1] / mat[0][0]
       ratio2 = mat[1][1] / mat[1][0]
       return iszero(ratio1 - ratio2)
   ```

   This does not take into account the possibility that one of the denominators could be zero.

Exercise 7.4. A plane is 2-dimensional, so it should have two elements in the basis. Note that the defining equation may be expressed as

$$\begin{bmatrix} 2 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and bringing the matrix on the left to row reduced form we get

$$A = \begin{bmatrix} 2 & -4 & -5 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -2 & -\frac{5}{2} \end{bmatrix} = R.$$

The nullspace of these matrices consists of precisely those vectors (x, y, z) which lie in the plane P. Note there are two free columns, so there are two special solutions. We find them quickly to be

$$\mathbf{s}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{s}_2 = \begin{bmatrix} \frac{5}{2} \\ 0 \\ 1 \end{bmatrix},$$

and get that the nullspace of the matrix A is the span of $\mathbf{s}_1$ and $\mathbf{s}_2$. Hence the plane P is the span of these equations. We did not check that $\{\mathbf{s}_1, \mathbf{s}_2\}$ is a linearly independent set, but because we know $\text{span}(\mathbf{s}_1, \mathbf{s}_2) = P$ and we know $\dim(P) = 2$, we must have that $\{\mathbf{s}_1, \mathbf{s}_2\}$ is linearly independent, because there are only two vectors in the set, and every basis of P must have 2 vectors. Hence $\{\mathbf{s}_1, \mathbf{s}_2\}$ is a basis for P.

Exercise 7.6. The proof of the first claim follows from first observing that the intersection $U \cap W$ is closed under vector addition and scalar multiplication. Indeed, if $\mathbf{v} \in U \cap W$, the $\mathbf{v} \in U$ (so $c\mathbf{v} \in U$) and $\mathbf{v} \in W$ (so $c\mathbf{v} \in W$). Hence $c\mathbf{v} \in U \cap W$. A similar appproach works for vector addition. The zero element is in both U and W, and so must be in $U \cap W$. Additive inverses are -1 multiples, and so are also in the intersection. The other properties are inherited from U and W similarly.

The proof of the second claim comes from constructing a basis for $U \cap W$ that can be extended to bases of U and W separately.

The proof of the third claim comes by constructing an explicit basis $\{(\mathbf{u}, 0) : \mathbf{u} \in B_U\} \cup \{(0, \mathbf{w}) : \mathbf{w} \in B_W\}$ for $V \oplus W$, where B_U is a basis for U and B_W is a basis for W.

Exercise 7.7.

1. The space of diagonal 3×3 matrices has the following basis (not the only one):

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\},$$

as any diagonal 3×3 matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ can be written as a linear combination of the given matrices. The space of skew-symmetric 3×3 matrices has the following basis (not the only one):

$$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\},$$

as any skew-symmetric 3×3 matrix $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ can be written as a linear combination of the given matrices.

2. The dimension of the space of $n \times n$ diagonal matrices is n, because the basis has elements only on the diagonal, and the diagonal has n elements.

   The basis of the space of $n \times n$ skew-symmetric matrices has as many elements as entries above (or below) the diagonal. The first row has $n-1$ entries above the diagonal, and every next row has one less. Hence the number of entries above the diagonal, and so the dimension, in general is

$$(n-1) + (n-2) + \cdots + 1 = \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}.$$

3. The identity and the identity with rows swapped are both invertible 2×2 matrices, yet their sum is not, as the sum only has one pivot:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

   Hence invertible 2×2 matrices are not closed under matrix addition, so cannot be a vector space.

   For the second part, we follow the hint and construct the basis of $\mathcal{M}_{2\times2}$ as linear combinations of invertible matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

   Hence the span of all invertible 2×2 matrices is all 2×2 matrices.

Exercise 8.1. To find these spaces, we have to row reduce the matrix and its transpose:

$$\text{rref}(A) = \begin{bmatrix} 0 & 1 & 0 & 0 & a+abc & abc-ab \\ 0 & 0 & 1 & 0 & -bc & b-bc \\ 0 & 0 & 0 & 1 & c & c \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \text{rref}(A^T) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the row-reduced form of A, it follows that

$$\mathrm{col}(A) = \mathrm{span}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}a\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\b\\1\\0\end{bmatrix}\right), \qquad \mathrm{null}(A) = \mathrm{span}\left(\begin{bmatrix}0\\-a-abc\\bc\\-c\\1\\0\end{bmatrix}, \begin{bmatrix}0\\ab-abc\\bc-b\\-c\\0\\1\end{bmatrix}\right).$$

From the row-reduced form of A^T, it follows that

$$\mathrm{row}(A) = \mathrm{span}\left(\begin{bmatrix}0\\1\\a\\0\\a\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\b\\0\\b\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\c\\c\end{bmatrix}\right), \qquad \mathrm{null}(A^T) = \mathrm{span}\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right).$$

If any of a, b, c are zero, then the spaces change accordingly. Their dimensions do not change. That is, no zero vectors appear, if any / all of a, b, c are zero.

Exercise 8.3. 1. The column space is the span of the columns, so all we need is to make $\mathbf{u}$ and $\mathbf{v}$ be two of the columns, and make the other two linear combinations of $\mathbf{u}$ and $\mathbf{v}$. There are many choices, one example is

$$A = \begin{bmatrix}1 & 3 & 0 & 0\\2 & 4 & 0 & 0\end{bmatrix}.$$

2. The given matrix is

$$A = \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3 & 4\end{bmatrix} + \left(\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3 & 4\end{bmatrix}\right)^2 = \begin{bmatrix}3 & 4\\6 & 8\end{bmatrix} + \begin{bmatrix}3 & 4\\6 & 8\end{bmatrix}\begin{bmatrix}3 & 4\\6 & 8\end{bmatrix} = \begin{bmatrix}3 & 4\\6 & 8\end{bmatrix} + \begin{bmatrix}33 & 44\\66 & 88\end{bmatrix} = \begin{bmatrix}36 & 48\\72 & 96\end{bmatrix}.$$

One row operation brings us to

$$\begin{bmatrix}36 & 48\\0 & 0\end{bmatrix},$$

so the column space is just the span of the first column. The row operation shows the second row is a multiple of the first, so the row space is the span of the first row. That is,

$$\mathrm{col}(A) = \mathrm{span}\left\{\begin{bmatrix}36\\72\end{bmatrix}\right\}, \qquad \mathrm{row}(A) = \mathrm{span}\left\{\begin{bmatrix}36\\48\end{bmatrix}\right\}.$$

Exercise 9.1. Arbitrary elements in U and V are

$$U \ni \mathbf{u} = a_1\mathbf{u}_1 + \cdots a_k\mathbf{u}_k = \sum_{i=1}^{k} a_i\mathbf{u}_i, \qquad V \ni \mathbf{v} = b_1\mathbf{v}_1 + \cdots b_\ell\mathbf{v}_\ell = \sum_{j=1}^{\ell} b_j\mathbf{v}_j.$$

Their dot product, following the laws of dot products, is

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (a_1\mathbf{u}_1 + \cdots a_k\mathbf{u}_k)\cdot(b_1\mathbf{v}_1 + \cdots b_\ell\mathbf{v}_\ell)\\ &= a_1b_1\mathbf{u}_1\cdot\mathbf{v}_1 + a_1b_2\mathbf{u}_1\cdot\mathbf{v}_2 + \cdots + a_kb_k\mathbf{u}_k\cdot\mathbf{v}_\ell\\ &= \sum_{i=1}^{k}\sum_{j=1}^{\ell} a_ib_j\underbrace{\mathbf{u}_i\cdot\mathbf{v}_j}_{0}\\ &= 0\end{aligned}$$

Exercise 9.3. Since the row space and the nullspace of A are orthogonal complements in $\mathbf{R}^n$, every

$\mathbf{x} \in \mathbf{R}^n$ can be decomposed as $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$, where $\mathbf{x}_r \in \text{row}(A)$ and $\mathbf{w}_n \in \text{null}(A)$. We claim that

$$\begin{aligned} f\colon \text{row}(A) &\to \text{col}(A), \\ \mathbf{v} &\mapsto A\mathbf{v} \end{aligned}$$

is such a function. We first note that $A\mathbf{v}$ is indeed in the column space of A, as

$$A\mathbf{v} = \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} | \\ \mathbf{a}_1 \\ | \end{bmatrix} + \cdots + v_n \begin{bmatrix} | \\ \mathbf{a}_n \\ | \end{bmatrix},$$

for $\mathbf{a}_1, \dots, \mathbf{a}_n$ the columns of A. To see that f is injective, suppose that $A\mathbf{v} = A\mathbf{w}$ for some $\mathbf{v}, \mathbf{w} \in \text{row}(A)$. Then

$$A\mathbf{v} - A\mathbf{w} = 0 \quad \Longrightarrow \quad A(\mathbf{v} - \mathbf{w}) = 0 \quad \Longrightarrow \quad \mathbf{v} - \mathbf{w} \in \text{null}(A).$$

However, since $\mathbf{v}, \mathbf{w} \in \text{row}(A)$, and $\text{row}(A)$ is a vector space, it follows that $\mathbf{v} - \mathbf{w} \in \text{row}(A)$ as well. Since $\text{row}(A)^\perp = \text{null}(A)$, the only vector in both $\text{row}(A)$ and $\text{null}(A)$ is the zero vector. Hence $\mathbf{v} - \mathbf{w} = 0$, or $\mathbf{v} = \mathbf{w}$.

To see f is surjective, let $\mathbf{c} \in \text{col}(A)$. That is, $\mathbf{c}$ is a linear combination of the columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ of A. In other words, there are real numbers $r_1, \dots, r_n$ with

$$\mathbf{c} = r_1\mathbf{a}_1 + \cdots + r_n\mathbf{a}_n = r_1 \begin{bmatrix} | \\ \mathbf{a}_1 \\ | \end{bmatrix} + \cdots + r_n \begin{bmatrix} | \\ \mathbf{a}_n \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}}_{\mathbf{r}}.$$

To see that $\mathbf{r} \in \text{row}(A)$, we again use the fact that $\text{row}(A)$ and $\text{null}(A)$ are orthogonal complements. That is, either $\mathbf{r}$ is in $\text{row}(A)$ or in $\text{null}(A)$, and if $\mathbf{r} \in \text{null}(A)$, then it follows that $A\mathbf{r} = \mathbf{c} = 0$. However, we also have $0 \in \text{row}(A)$, and $A0 = 0 = \mathbf{c}$ as well, so $\mathbf{r}$ has a preimage in $\text{row}(A)$. Alternatively, if $\mathbf{r} \in \text{row}(A)$, then we are done.

Hence f is a bijection.

Exercise 9.4.

1. Let $\mathbf{u} \in U$, $\mathbf{u}' \in U^\perp$, and $u'' \in (U^\perp)^\perp$. To see that $U \subseteq (U^\perp)^\perp$, notice that $\mathbf{u} \cdot \mathbf{u}' = 0$, which means that $u \in (U^\perp)^\perp)$. To see that $(U^\perp)^\perp \subseteq U$, notice that $u'' \notin U^\perp$, and since $U \perp +U = \mathbf{R}^n$, it must be that $u'' \in U$. Hence $U = (U^\perp)^\perp$.

2. For this question we use Theorem 7.9 from the lecture notes, which showed that $(U + V)^\perp = U^\perp \cap V^\perp$. In this statement, replace U with $U^\perp$ and V with $V^\perp$:

$$\begin{aligned} (U + V)^\perp &= U^\perp \cap V^\perp && \text{(Theorem 7.9, part 2)} \\ (U^\perp + V^\perp)^\perp &= (U^\perp)^\perp \cap (V^\perp)^\perp && \text{(replace } U \text{ with } U^\perp \text{ and } V \text{ with } V^\perp\text{)} \\ (U^\perp + V^\perp)^\perp &= U \cap V && \text{(part (a) to this question)} \\ ((U^\perp + V^\perp)^\perp)^\perp &= (U \cap V)^\perp && \text{(take the complement of both sides)} \\ U^\perp + V^\perp &= (U \cap V)^\perp && \text{(part (a) to this question)} \end{aligned}$$

3. The nullspace is orthogonal to the rowspace by Example 7.10, so for such a matrix C, we need $\text{row}(C) = U + V$. That is, if we just put the columns of A and B as rows of C, we will have the desired matrix:

$$C = \begin{bmatrix} A^T \\ B^T \end{bmatrix}.$$

Exercise 9.5. 1. Normal vectors can be read off from the equations, and they are

$$\mathbf{n}_1 = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, \qquad \mathbf{n}_2 = \begin{bmatrix} 5 \\ 0 \\ -10 \end{bmatrix}.$$

2. We follow the hint, and starting with P_1, describe the equation as a matrix equation

$$\begin{bmatrix} 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

The points in the plane are elements in the nullspace of the matrix

$$\begin{aligned} A = \begin{bmatrix} 3 & -4 & 1 \end{bmatrix} &\xrightarrow{RREF} \begin{bmatrix} 1 & -4/3 & 1/3 \end{bmatrix} \\ &\implies \text{null}(A) = \text{span}\left\{ \begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} \right\} \\ &\implies B_1 = \left\{ \begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

Similarly for the plane P_2, we find

$$\begin{aligned} A = \begin{bmatrix} 5 & 0 & -10 \end{bmatrix} &\xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \\ &\implies \text{null}(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} \\ &\implies B_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

3. Here we can simply use the basis vectors of P_1 as rows:

$$A_1 = \begin{bmatrix} 4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix}.$$

We see the nullspace immediately as $\text{null}(A_1) = \text{span}\left\{ \left[\begin{smallmatrix} 3 \\ -4 \\ 1 \end{smallmatrix} \right] \right\} = \text{span}\{\mathbf{n}_1\}$.

4. As above, we use the basis vectors of P_2 as columns:

$$A_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The left nullspace is the nullspace of the transpose:

$$A_2^T = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}. \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

We see the nullspace immediately as

$$\text{null}(A_2^T) = \text{span}\left\{ \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ -\frac{1}{10}\mathbf{n}_2 \right\} = \text{span}\{\mathbf{n}_2\}.$$

Exercise 10.1. We expand the expression P^2 to get

$$P^2 = \left(\frac{1}{\mathbf{v}\bullet\mathbf{v}}\cdot\mathbf{v}\mathbf{v}^T\right)\cdot\left(\frac{1}{\mathbf{v}\bullet\mathbf{v}}\cdot\mathbf{v}\mathbf{v}^T\right) = \frac{(\mathbf{v}\mathbf{v}^T)(\mathbf{v}\mathbf{v}^T)}{(\mathbf{v}\bullet\mathbf{v})(\mathbf{v}\bullet\mathbf{v})} = \frac{\mathbf{v}(\mathbf{v}^T\mathbf{v})\mathbf{v}^T}{(\mathbf{v}\bullet\mathbf{v})(\mathbf{v}\bullet\mathbf{v})} = \frac{\mathbf{v}(\mathbf{v}\bullet\mathbf{v})\mathbf{v}^T}{(\mathbf{v}\bullet\mathbf{v})(\mathbf{v}\bullet\mathbf{v})} = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}\bullet\mathbf{v}} = P,$$

as desired.

Exercise 10.3. The normal vector is $\mathbf{n} = (3, 4, 9)$. Following Exercise 7.4, we find the plane $3x+4y-9z$ to be the column space of the matrix

$$A = \begin{bmatrix} -\frac{4}{3} & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Exercise 10.4.

1. The last two coordinates disappear, so we are looking for a block matrix

$$M = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. All the rows are moving forward, so we need a matrix with rows like the identity matrix, but also moved forward (up). That is, we need

$$N = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Exercise 10.5.

1. Recall that the projection matrix for proj_U is $A(A^TA)^{-1}A^T$, where the columns of A are the $\mathbf{u}_i$. By Remark 8.6, the projection matrix for $\text{proj}_{U^\perp}$ is $I - A(A^TA)^{-1}A^T$. Hence

$$\text{refl}_U(\mathbf{v}) = \mathbf{v} - 2\left(I - A(A^TA)^{-1}A^T\right)\mathbf{v} = \left(I - 2 + 2A(A^TA)^{-1}A^T\right)\mathbf{v},$$

and so the matrix is $\text{refl}_U = 2A(A^TA)^{-1}A^T - I$.

2. This comes from a straightforward computation:

$$\begin{aligned}
\|\text{refl}_U(\mathbf{v})\|^2 &= \|\left(2A(A^TA)^{-1}A^T - I\right)\mathbf{v}\|^2 \\
&= \|2A(A^TA)^{-1}A^T\mathbf{v} - \mathbf{v}\|^2 \\
&= \left(2A(A^TA)^{-1}A^T\mathbf{v} - \mathbf{v}\right)\cdot\left(2A(A^TA)^{-1}A^T\mathbf{v} - \mathbf{v}\right) \\
&= (2A(A^TA)^{-1}A^T\mathbf{v})^T(2A(A^TA)^{-1}A^T\mathbf{v}) - 2(2A(A^TA)^{-1}A^T\mathbf{v})^T\mathbf{v} + \mathbf{v}^T\mathbf{v} \\
&= 4\mathbf{v}^TA((A^TA)^{-1})^TA^TA(A^TA)^{-1}A^T\mathbf{v} - 4\mathbf{v}^TA((A^TA)^{-1})^TA^T\mathbf{v} + \|\mathbf{v}\|^2 \\
&= 4\mathbf{v}^TA(A^TA)^{-1}A^TA(A^TA)^{-1}A^T\mathbf{v} - 4\mathbf{v}^TA(A^TA)^{-1}A^T\mathbf{v} + \|\mathbf{v}\|^2 \\
&= 4\mathbf{v}^TA(A^TA)^{-1}A^T\mathbf{v} - 4\mathbf{v}^TA(A^TA)^{-1}A^T\mathbf{v} + \|\mathbf{v}\|^2 \\
&= \|\mathbf{v}\|^2.
\end{aligned}$$

We used the fact that the transpose of the inverse is the inverse of the tranpose: $(A^{-1})^T = (A^T)^{-1}$, which follows from taking the inverse of $AA^{-1} = I$.

Exercise 11.2. Recal the nullspace of A is all the vectors $\mathbf{x}$ for which $A\mathbf{x} = 0$. To see $\text{null}(A) \subseteq \text{null}(A^TA)$, suppose that $\mathbf{x} \in \text{null}(A)$. That is, $A\mathbf{x} = 0$, and multiplying by A^T on the left gives $A^TA\mathbf{x} = 0$, which means $\mathbf{x} \in \text{null}(A^TA)$. To see $\text{null}(A^TA) \subseteq \text{null}(A)$, suppose that $\mathbf{y} \in \text{null}(A^TA)$. That is, $A^TA\mathbf{y} = 0$, and multiplying by $\mathbf{y}^T$ on the left gives

$$0 = \mathbf{y}^T(A^TA\mathbf{y}) = (\mathbf{y}^TA^T)(A\mathbf{y}) = (A\mathbf{y})^T(A\mathbf{y}) = \|A\mathbf{y}\|.$$

Since the norm is positive definite, it follow that $A\mathbf{y} = 0$, and so $\mathbf{y} \in \text{null}(A)$.

Exercise 11.3. 1. We are trying to find the best solution $\mathbf{x}$ to the equation

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 4 & 1 \\ 3 & 1 \\ -2 & 1 \\ 6 & 1 \\ -6 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 1 \\ -3 \\ -7 \\ 4 \end{bmatrix}}_{\mathbf{b}},$$

or equivalently, trying to find the projection of $\mathbf{b}$ onto $\text{col}(A)$. That is, we can either solve $A\mathbf{x} = A(A^TA)^{-1}A^T\mathbf{b}$, or solve $A^TA\mathbf{x} = A^T\mathbf{b}$:

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 4 & 1 \\ 3 & 1 \\ -2 & 1 \\ 6 & 1 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}}_{A\mathbf{x}} = \underbrace{\frac{1}{596} \begin{bmatrix} 784 \\ -376 \\ -144 \\ 1016 \\ -840 \\ 1944 \end{bmatrix}}_{A(A^TA)^{-1}A^T\mathbf{b}} \qquad \text{or} \qquad \underbrace{\begin{bmatrix} 102 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}}_{A^TA\mathbf{x}} = \underbrace{\begin{bmatrix} -36 \\ 4 \end{bmatrix}}_{A^T\mathbf{b}}.$$

The second is easier to solve:

$$\begin{bmatrix} 102 & 4 & -36 \\ 4 & 6 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -\frac{58}{149} \\ 0 & 1 & \frac{138}{149} \end{bmatrix},$$

and so the equation of the line is $y = -\frac{58}{149}x + \frac{138}{149}$.

2. Using the derivatives approach, we know that that the line $y = -\frac{58}{149}x + \frac{138}{149}$ minimizes the squares of the vertical distances between every point in P and the line. That is, choosing p_7 to be on the line will keep the sum of squares at this minimum. The are many such choices, one of which is

$$p_7 = \left(0, \frac{138}{149}\right).$$

3. This is solved in reverse, by adding a point (z, w) to the process, and running the same steps as above. We are trying to solve

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 4 & 1 \\ 3 & 1 \\ -2 & 1 \\ 6 & 1 \\ -6 & 1 \\ z & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 1 \\ -3 \\ -7 \\ 4 \\ w \end{bmatrix}}_{\mathbf{b}},$$

or equivalently,

$$A^TA \begin{bmatrix} a \\ b \end{bmatrix} = A^T\mathbf{b} \iff \begin{bmatrix} 102 + z^2 & 4 + z \\ 4 + z & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -36 + wz \\ 4 + w \end{bmatrix}.$$

We row reduce the augmented matrix:

$$\begin{bmatrix} 102 + z^2 & 4 + z & -36 + wz \\ 4 + z & 7 & 4 + w \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & \frac{3wz - 2w - 2z - 134}{3z^2 - 4z + 349} \\ 0 & 1 & \frac{-2wz + 51w + 2z^2 + 18z + 276}{3z^2 - 4z + 349} \end{bmatrix}.$$

The slope is the entry on the top right, so we simply need some values z, w for which

$$c = \frac{3wz - 2w - 2z - 134}{3z^2 - 4z + 349}.$$

There are many solutions (z, w), so we just let $z = 0$ and solve

$$c = \frac{-2w - 134}{349} \iff w = \frac{-349c - 134}{2}.$$

Hence adding the point $\left(0, \frac{-349c-134}{2}\right)$ will make the least squares approximation have slope c.

Exercise 11.4. There are many ways to do this, one way is given below.

```
import numpy as np
from numpy.linalg import solve

def degreedlsq(points,degree):
    n = len(points)

    # Construct the Vandermonde matrix
    matrix = np.ones((n,degree+1))
    for row,point in enumerate(points):
        for column in range(1,degree+1):
            matrix[row][column] = point[0]**column

    # Set up the least squares equation and solve it
    ATA = np.matmul(np.transpose(matrix),matrix)
    ATb = np.matmul(np.transpose(matrix),np.transpose(points)[1])
    solution = solve(ATA,ATb)

    # Print the result in a nice way
    result = str(round(solution[0],2))
    for i,coefficient in enumerate(solution[1:]):
        result = str(round(coefficient,2))+"x^"+str(i+1)+" + "+result
    print(result)
```

Exercise 11.5. 1. Recall the equation of a plane is $ax + by + cz = d$, for some $a, b, c, d \in \mathbf{R}$. Since we are told the plane will not go through the origin, $d \neq 0$, so we can divide by d and just consider the equation $ax + by + cz = 1$. The system of equation we are trying to solve is then

$$\begin{aligned} a - 2b - 4c &= 1 \\ 5b + 5c &= 1 \\ -6a - 7b + 2c &= 1 \\ a + 4b - c &= 1 \end{aligned} \iff \underbrace{\begin{bmatrix} 1 & -2 & -4 \\ 0 & 5 & 5 \\ -6 & -7 & 2 \\ 1 & 4 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}}.$$

Having four equations in three unknowns is overdetermined, so we need to project to solve it. We can either solve $A\mathbf{x} = A(A^TA)^{-1}A^T\mathbf{b}$, or equivalently, solve $A^TA\mathbf{x} = A^T\mathbf{b}$. The second of these equations becomes

$$A^TA\mathbf{x} = \begin{bmatrix} 38 & 44 & -17 \\ 44 & 94 & 15 \\ -17 & 15 & 46 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix} = A^T\mathbf{b}.$$

Bringing the augmented matrix to reduced row echelon form we see

$$\begin{bmatrix} A^TA\mathbf{x} & A^T\mathbf{b} \end{bmatrix} = \begin{bmatrix} 38 & 44 & -17 & -4 \\ 44 & 94 & 15 & 0 \\ -17 & 15 & 46 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -66/95 \\ 0 & 1 & 0 & 36/95 \\ 0 & 0 & 1 & -32/95 \end{bmatrix},$$

and hence the equation of the closest plane is $-66x + 36y - 32z = 95$.

2. Recall that projection works to subspaces, and having $95 \neq 0$ means the plane H is not a subspace. Our approach will be to move H so that it is a subspace, then project, then move H back. Note that the point $\mathbf{w} = (-\frac{95}{66}, 0, 0)$ lies in H, so we shift everything in H by $\mathbf{w}$:

$$\begin{aligned} H &= \{(x, y, z) \ : \ -66x + 36y - 32z = 95\}, \\ H' &= \{(x + \tfrac{95}{66}, y, z) \ : \ -66x + 36y - 32z = 95\} \\ &= \{(x, y, z) \ : \ -66x + 36y - 32z = 0\}. \end{aligned}$$

As before, we find the basis of H' from the nullspace of

$$\begin{bmatrix} -66 & 36 & -32 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & -\frac{6}{11} & \frac{16}{33} \end{bmatrix},$$

which is the span of $\left[\begin{smallmatrix} 6/11 \\ 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -16/33 \\ 0 \\ 1 \end{smallmatrix}\right]$, which we can then use as columns of the matrix A:

$$A = \begin{bmatrix} \frac{6}{11} & -\frac{16}{33} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \implies P = A(A^TA)^{-1}A^T = \frac{1}{1669} \begin{bmatrix} 580 & 594 & -528 \\ 594 & 1345 & 288 \\ -528 & 288 & 1413 \end{bmatrix}.$$

The points we project to H' are not p_1, p_2, p_3, p_4, but rather $p_1 - \mathbf{w}$, $p_2 - \mathbf{w}$, $p_3 - \mathbf{w}$ and $p_4 - \mathbf{w}$:

$$P(p_1 - \mathbf{w}) = \frac{1}{1669} \begin{bmatrix} 580 & 594 & -528 \\ 594 & 1345 & 288 \\ -528 & 288 & 1413 \end{bmatrix} \begin{bmatrix} 1 - \frac{95}{66} \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{77182}{55077} \\ -\frac{2393}{1669} \\ -\frac{7516}{1669} \end{bmatrix} = \mathbf{q}_1$$

$$P(p_2 - \mathbf{w}) = \frac{1}{1669} \begin{bmatrix} 580 & 594 & -528 \\ 594 & 1345 & 288 \\ -528 & 288 & 1413 \end{bmatrix} \begin{bmatrix} \frac{95}{66} \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{38440}{55077} \\ \frac{9020}{1669} \\ \frac{7745}{1669} \end{bmatrix} = \mathbf{q}_2$$

$$P(p_3 - \mathbf{w}) = \frac{1}{1669} \begin{bmatrix} 580 & 594 & -528 \\ 594 & 1345 & 288 \\ -528 & 288 & 1413 \end{bmatrix} \begin{bmatrix} -6 + \frac{95}{66} \\ -7 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{259352}{55077} \\ -\frac{11548}{1669} \\ \frac{3218}{1669} \end{bmatrix} = \mathbf{q}_3$$

$$P(p_4 - \mathbf{w}) = \frac{1}{1669} \begin{bmatrix} 580 & 594 & -528 \\ 594 & 1345 & 288 \\ -528 & 288 & 1413 \end{bmatrix} \begin{bmatrix} 1 - \frac{95}{66} \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{145522}{55077} \\ \frac{6541}{1669} \\ \frac{1549}{1669} \end{bmatrix} = \mathbf{q}_4$$

These points lie on H'. We need to add $\mathbf{w}$ to them to make sure they lie on H. That is,

$$\begin{aligned} \text{proj}_H(p_1) &= \mathbf{q}_1 + \mathbf{w} \\ \text{proj}_H(p_2) &= \mathbf{q}_2 + \mathbf{w} \\ \text{proj}_H(p_3) &= \mathbf{q}_3 + \mathbf{w} \\ \text{proj}_H(p_4) &= \mathbf{q}_4 + \mathbf{w} \end{aligned}$$

The expressions are too long to calculate, so we leave the vectors as they are above.

Exercise 12.2.

- **Step 1:** Set $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$.

- **Step 2:** Project $\mathbf{v}_2$ onto $\mathbf{w}_1$, and subtract this from $\mathbf{v}_2$ to ensure the new vector will be orthogonal to the previous vector. That is, set $\mathbf{w}_2$ to be the error vector when projecting to $\mathbf{w}_1$. Using the formula from Definition 10.2, we get

$$\mathbf{w}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{w}_1}(\mathbf{v}_2) = \mathbf{v}_2 - \frac{\mathbf{w}_1^T\mathbf{v}_2}{\mathbf{w}_1^T\mathbf{w}_1}\mathbf{w}_1 = \begin{bmatrix}2\\0\\2\\0\end{bmatrix} - \frac{2}{6}\begin{bmatrix}1\\2\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{5}{3}\\-\frac{2}{3}\\2\\-\frac{1}{3}\end{bmatrix}$$

- **Step 3:** Project $\mathbf{v}_3$ onto $\mathbf{w}_1$ and $\mathbf{w}_2$, and subtract these from $\mathbf{v}_3$ to make sure everything is still orthogonal. The formula is

$$\mathbf{w}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{w}_1}(\mathbf{v}_3) - \text{proj}_{\mathbf{w}_2}(\mathbf{v}_3) = \mathbf{v}_3 - \frac{\mathbf{w}_1^T\mathbf{v}_3}{\mathbf{w}_1^T\mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{w}_2^T\mathbf{v}_3}{\mathbf{w}_2^T\mathbf{w}_2}\mathbf{w}_2 = \begin{bmatrix}-\frac{8}{11}\\\frac{1}{11}\\\frac{8}{11}\\\frac{6}{11}\end{bmatrix}.$$

- **Step 4:** Repeat the same for $\mathbf{v}_4$ to get

$$\mathbf{w}_4 = \mathbf{v}_4 - \text{proj}_{\mathbf{w}_1}(\mathbf{v}_4) - \text{proj}_{\mathbf{w}_2}(\mathbf{v}_4) - \text{proj}_{\mathbf{w}_3}(\mathbf{v}_4) = \mathbf{v}_4 - \frac{\mathbf{w}_1^T\mathbf{v}_4}{\mathbf{w}_1^T\mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{w}_2^T\mathbf{v}_4}{\mathbf{w}_2^T\mathbf{w}_2}\mathbf{w}_2 - \frac{\mathbf{w}_3^T\mathbf{v}_4}{\mathbf{w}_3^T\mathbf{w}_3}\mathbf{w}_3 = \begin{bmatrix}\frac{1}{3}\\-\frac{2}{3}\\-\frac{1}{3}\\1\end{bmatrix}.$$

We now have an orthogonal basis of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for $\mathbf{R}^4$. Note these do not (except for the first one) point in the same directions as the original set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, but they do have the same span. Normalizing these vectors we get the final set:

$$\mathbf{q}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{bmatrix}1/\sqrt{6}\\\sqrt{2}/\sqrt{6}\\0\\1/\sqrt{6}\end{bmatrix}, \qquad \mathbf{q}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{bmatrix}5/\sqrt{66}\\-\sqrt{2}/\sqrt{33}\\\sqrt{6}/\sqrt{11}\\-1/\sqrt{66}\end{bmatrix},$$

$$\mathbf{q}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \begin{bmatrix}-8/\sqrt{165}\\1/\sqrt{165}\\8/\sqrt{165}\\\sqrt{12}/\sqrt{165}\end{bmatrix}, \qquad \mathbf{q}_4 = \frac{\mathbf{w}_4}{\|\mathbf{w}_4\|} = \begin{bmatrix}1/\sqrt{15}\\-2/\sqrt{15}\\-1/\sqrt{15}\\\sqrt{3}/\sqrt{5}\end{bmatrix}.$$

Exercise 18.1. For the matrix A, note that the rows are multiples of each other, so $\lambda_1 = 0$. Since there are 2 eigenvalues (as it is a 2×2 matrix), and the sum of the eigenvalues is the trace, it follows that $\lambda_1 + \lambda_2 = 2 + 5 = 7$, so $\lambda_2 = 7$. For the eigenvectors, we eliminate the augmented matrices

$$\begin{bmatrix}2 & 2 & 0\\5 & 5 & 0\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 1 & 0\\0 & 0 & 0\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}-5 & 2 & 0\\5 & -2 & 0\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & -2/5 & 0\\0 & 0 & 0\end{bmatrix}.$$

So the eigenvectors are $\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$ for $\lambda_1 = 0$ and $\left[\begin{smallmatrix}2/5\\1\end{smallmatrix}\right]$ for $\lambda_2 = 7$, giving the decomposition

$$\begin{bmatrix}2 & 2\\5 & 5\end{bmatrix} = \begin{bmatrix}-1 & 2/5\\1 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 7\end{bmatrix}\begin{bmatrix}-1 & 2/5\\1 & 1\end{bmatrix}^{-1} \quad \text{where} \quad \begin{bmatrix}-1 & 2/5\\1 & 1\end{bmatrix}^{-1} = \frac{-5}{7}\begin{bmatrix}1 & -2/5\\-1 & -1\end{bmatrix}.$$

For the matrix B, the eigenvalues are on the diagonal, but the eigenvectors are not so immediate. For

$\lambda_1 = 1$ we have $\mathbf{e}_1$, but for $\lambda_2 = 4$ we need

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4x \\ 4y \\ 4z \end{bmatrix} \implies \begin{aligned} z &= 0 \\ 4y &= 4y \\ -3x &= -2y \end{aligned} \implies \mathbf{v}_2 = \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}.$$

Similarly for $\lambda_3 = 6$, we need

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6x \\ 6y \\ 6z \end{bmatrix} \implies \begin{aligned} 6z &= 6z \\ -2y &= -5z \\ -5x &= -2y - 3z \end{aligned} \implies \mathbf{v}_3 = \begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix}.$$

Hence the decomposition is

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2/3 & 8/5 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2/3 & 8/5 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{bmatrix}^{-1},$$

where

$$\begin{bmatrix} 1 & 2/3 & 8/5 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2/3 & 1/15 \\ 0 & 1 & -5/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise 18.2. 1. The eigenvector matrix X has the eigenvectors as columns, and the eigenvalues matrix Λ has the eigenvalues on the diagonal:

$$X = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

2. First we get the inverse of X by row reduction:

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}.$$

Hence the matrix A is

$$\begin{aligned} X\Lambda X^{-1} &= \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} X^{-1} \\ &= \begin{bmatrix} -1 & 0 & 3 \\ -2 & 2 & 3 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 0 & 2 \\ -5 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix}. \end{aligned}$$

Exercise 18.4. 1. This follows by computing the diagonal entries of AB and of BA. By the formula for entries in a product of matrices:

$$(AB)_{ii} = \sum_{j=1}^{3} A_{ij}B_{ji} = A_{i1}B_{1i} + A_{i2}B_{2i} + A_{i3}B_{3i},$$

$$(BA)_{ii} = \sum_{j=1}^{3} B_{ij}A_{ji} = B_{i1}A_{1i} + B_{i2}A_{2i} + B_{i3}A_{3i}.$$

Summing up for $i = 1, 2, 3$, for the trace, we find that

$$\begin{aligned}
\text{trace}(AB) &= (AB)_{11} + (AB)_{22} + (AB)_{33} \\
&= A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} + A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} + A_{31}B_{13} + A_{32}B_{23} + A_{33}B_{33}, \\
\text{trace}(BA) &= (BA)_{11} + (BA)_{22} + (BA)_{33} \\
&= B_{11}A_{11} + B_{12}A_{21} + B_{13}A_{31} + B_{21}A_{12} + B_{22}A_{22} + B_{23}A_{32} + B_{31}A_{13} + B_{32}A_{23} + B_{33}A_{33},
\end{aligned}$$

which are the same.

2. Since C is diagonalizable, there exists an invertible matrix X of the eigenvectors of C as columns, and a diagonal matrix Λ of the eigenvalues of C on its diagonal, with $C = X\Lambda X^{-1}$. Using the previous task with $A = X$ and $B = \Lambda X^{-1}$, we get that

$$\begin{aligned}
\text{trace}(C) &= \text{trace}(X\Lambda X^{-1}) && \text{(since } C \text{ is diagonalizable)} \\
&= \text{trace}((A)(\Lambda X^{-1})) \\
&= \text{trace}((\Lambda X^{-1})(X)) && \text{(by part (a) above)} \\
&= \text{trace}(\Lambda X^{-1}X) \\
&= \text{trace}(\Lambda I) && \text{(definition of the inverse)} \\
&= \text{trace}(\Lambda) \\
&= \Lambda_{11} + \Lambda_{22} + \Lambda_{33}. && \text{(since } \Lambda \text{ is diagonal)}
\end{aligned}$$

This is the sum of the eigenvalues of C, since the eigenvalues of C are on the diagonal of Λ.

3. If the eigenvalues of C are $1, \frac{1}{2}, \frac{1}{3}$, then the eigenvalues of C^n are $1^n, \frac{1}{2^n}, \frac{1}{3^n}$, as

$$\begin{aligned}
C^2 &= (X\Lambda X^{-1})(X\Lambda X^{-1}) = X\Lambda I\Lambda X^{-1} = X\Lambda^2 X^{-1} \\
C^3 &= (X\Lambda^2 X^{-1})(X\Lambda X^{-1}) = X\Lambda^2 I\Lambda X^{-1} = X\Lambda^3 X^{-1} \\
&\vdots \\
C^n &= X\Lambda^n X^{-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{n\to\infty} C^n &= \lim_{n\to\infty} \left(X\Lambda^n X^{-1}\right) \\
&= \lim_{n\to\infty} \left(\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} 1^n & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{3^n} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \right) \\
&= \lim_{n\to\infty} \left(\begin{bmatrix} x_{11} & \frac{x_{12}}{2^n} & \frac{x_{13}}{3^n} \\ x_{21} & \frac{x_{22}}{2^n} & \frac{x_{23}}{3^n} \\ x_{31} & \frac{x_{32}}{2^n} & \frac{x_{33}}{3^n} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \right) \\
&= \lim_{n\to\infty} \left(\begin{bmatrix} x_{11}y_{11}+\frac{1}{2^n}\cdot x_{12}y_{21}+\frac{1}{3^n}\cdot x_{13}y_{31} & x_{11}y_{12}+\frac{1}{2^n}\cdot x_{12}y_{22}+\frac{1}{3^n}\cdot x_{13}y_{32} & x_{11}y_{13}+\frac{1}{2^n}\cdot x_{12}y_{23}+\frac{1}{3^n}\cdot x_{13}y_{33} \\ x_{21}y_{11}+\frac{1}{2^n}\cdot x_{22}y_{21}+\frac{1}{3^n}\cdot x_{23}y_{31} & x_{21}y_{12}+\frac{1}{2^n}\cdot x_{22}y_{22}+\frac{1}{3^n}\cdot x_{23}y_{32} & x_{21}y_{13}+\frac{1}{2^n}\cdot x_{22}y_{23}+\frac{1}{3^n}\cdot x_{23}y_{33} \\ x_{31}y_{11}+\frac{1}{2^n}\cdot x_{32}y_{21}+\frac{1}{3^n}\cdot x_{33}y_{31} & x_{31}y_{12}+\frac{1}{2^n}\cdot x_{32}y_{22}+\frac{1}{3^n}\cdot x_{33}y_{32} & x_{31}y_{13}+\frac{1}{2^n}\cdot x_{32}y_{23}+\frac{1}{3^n}\cdot x_{33}y_{33} \end{bmatrix} \right) \\
&= \begin{bmatrix} x_{11}y_{11} & x_{11}y_{12} & x_{11}y_{13} \\ x_{21}y_{11} & x_{21}y_{12} & x_{21}y_{13} \\ x_{31}y_{11} & x_{31}y_{12} & x_{31}y_{13} \end{bmatrix} \\
&= \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \end{bmatrix}.
\end{aligned}$$

The limit exists, and as it is a (outer) product of two vectors, it must have rank at most 1. Moreover, we know it has rank exactly one, because to have rank zero either the column or row vector must be all zeros - but this is not possible, as then the matrices X, X^{-1} would not be invertible.

Exercise 19.1. We apply equation (9) from Remark 20.4 above:

$$(A-6I)\mathbf{v}_2 = \mathbf{v}_1 \quad \Longleftrightarrow \quad \begin{bmatrix} 3 & -1 & -1 & -3 \\ -3 & -1 & 1 & 1 \\ 5 & -5 & -1 & -9 \\ 3 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Longleftrightarrow \quad \begin{bmatrix} 3 & -1 & -1 & -3 & 1 \\ -3 & -1 & 1 & 1 & -1 \\ 5 & -5 & -1 & -9 & 1 \\ 3 & 1 & -1 & -1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and so $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T$. We similarly solve $(A-6I)\mathbf{v}_3 = \mathbf{v}_2$ and $(A-6I)\mathbf{v}_4 = \mathbf{v}_3$ to get the matrix $B \in \mathcal{M}_{4\times 4}$, which has the generalized eigenvectors as its columns. Moreover, we notice that

$$B^{-1}AB = \underbrace{\begin{bmatrix} -3 & -1 & 1 & 2 \\ 5 & 1 & -1 & -3 \\ -2 & -4 & 0 & -2 \\ 0 & 4 & 0 & 4 \end{bmatrix}}_{B^{-1}} \underbrace{\begin{bmatrix} 9 & -1 & -1 & -3 \\ -3 & 5 & 1 & 1 \\ 5 & -5 & 5 & -9 \\ 3 & 1 & -1 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & -1 & -\frac{3}{4} \\ 1 & 0 & -\frac{3}{2} & -\frac{5}{4} \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{B} = \begin{bmatrix} 6 & 1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix},$$

which is the Jordan form of A.

Exercise 20.1. We apply equation (9) from Remark 20.4 above:

$$(A-6I)\mathbf{v}_2 = \mathbf{v}_1 \quad \Longleftrightarrow \quad \begin{bmatrix} 3 & -1 & -1 & -3 \\ -3 & -1 & 1 & 1 \\ 5 & -5 & -1 & -9 \\ 3 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Longleftrightarrow \quad \begin{bmatrix} 3 & -1 & -1 & -3 & 1 \\ -3 & -1 & 1 & 1 & -1 \\ 5 & -5 & -1 & -9 & 1 \\ 3 & 1 & -1 & -1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and so $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T$. We similarly solve $(A-6I)\mathbf{v}_3 = \mathbf{v}_2$ and $(A-6I)\mathbf{v}_4 = \mathbf{v}_3$ to get the matrix $B \in \mathcal{M}_{4\times 4}$, which has the generalized eigenvectors as its columns. Moreover, we notice that

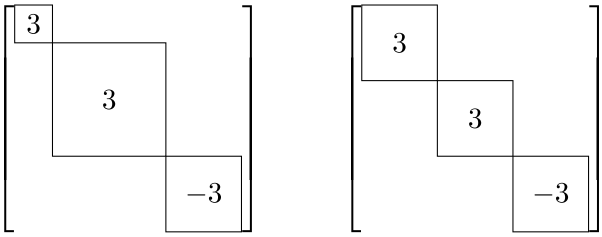
$$B^{-1}AB = \underbrace{\begin{bmatrix} -3 & -1 & 1 & 2 \\ 5 & 1 & -1 & -3 \\ -2 & -4 & 0 & -2 \\ 0 & 4 & 0 & 4 \end{bmatrix}}_{B^{-1}} \underbrace{\begin{bmatrix} 9 & -1 & -1 & -3 \\ -3 & 5 & 1 & 1 \\ 5 & -5 & 5 & -9 \\ 3 & 1 & -1 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & -1 & -\frac{3}{4} \\ 1 & 0 & -\frac{3}{2} & -\frac{5}{4} \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{B} = \begin{bmatrix} 6 & 1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix},$$

which is the Jordan form of A.

Exercise 20.3. 1. The eigenvalue -3 contributes a 2×2 Jordan block, since the algebraic multiplicity is 2 (so all its Jordan blocks together have 2 rows and 2 columns) and the geometric multiplicity is 1 (so there is only one Jordan block corresponding to this eigenvalue). Similarly, the Jordan blocks for the eigenvalue 3 take up 4 rows and 4 columns, and there are 2 of them. Hence:

- there are 3 Jordan blocks

- their sizes are either 1,3,2 or 2,2,2:



The order of the blocks is not relevant for this question.

2. For the matrix B, we need to find an invertible 6×6 matrix C for which $B = CJC^{-1}$, as the J and B will be similar. We need B to have no zero entries, and generating several random matrices with entries in the range $\{-1, 0, 1\}$, we quickly find one (there is not a unique answer). We see that

$$B = \underbrace{\begin{bmatrix} -1 & -1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} 2 & 1 & 2 & 3 & 1 & -1 \\ -2 & -1 & -1 & -2 & -1 & 2 \\ -3 & -2 & -2 & -3 & -1 & 3 \\ 4 & 2 & 3 & 5 & 1 & -2 \\ -3 & -2 & -2 & -4 & -1 & 2 \\ 3 & 2 & 2 & 3 & 1 & -2 \end{bmatrix}}_{C^{-1}} = \begin{bmatrix} 12 & 5 & 6 & 16 & 3 & -6 \\ -39 & -23 & -26 & -45 & -13 & 26 \\ 15 & 11 & 13 & 20 & 5 & -10 \\ -3 & -2 & -2 & -6 & -1 & 2 \\ -16 & -11 & -10 & -15 & -3 & 12 \\ -14 & -10 & -9 & -13 & -5 & 13 \end{bmatrix}.$$

3. Applying Theorem 20.10 from the lecture notes and the fact that $J = C^{-1}BC$, we get that the generalized eigenvectors of B are the columns of C.

Exercise 21.1. 1. There are 50 entries in the matrix, but they repeat horizontally. We could do SVD, but we can see the decomposition by sight:

$$L = \mathbf{u}\mathbf{v}^T = \begin{bmatrix} r \\ r \\ w \\ r \\ r \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

So now instead of having 5×10 pieces of data, we have only $5 + 10$, a 70% reduction in size.

2. We can reduce the 7×9 pieces of data by singular value decomposition. For ease of notation, change r to 1 and w to 0. We then simply compute the eigenvalues and eigenvectors of WW^T and W^TW. We are lucky and see there are only 2 nonzero eigenvalues:

$$(\sigma_1^2, \sigma_2^2) \approx (18.93, 5.07), \qquad \mathbf{u}_1 \approx \begin{bmatrix} -0.23 \\ -0.23 \\ -0.63 \\ 0 \\ -0.63 \\ -0.23 \\ -0.23 \end{bmatrix}, \mathbf{u}_2 \approx \begin{bmatrix} -0.44 \\ -0.44 \\ 0.33 \\ 0 \\ 0.33 \\ -0.44 \\ -0.44 \end{bmatrix}, \qquad \mathbf{v}_1 \approx \begin{bmatrix} -0.29 \\ -0.29 \\ -0.29 \\ -0.5 \\ 0 \\ -0.5 \\ -0.29 \\ -0.29 \\ -0.29 \end{bmatrix}, \mathbf{v}_2 \approx \begin{bmatrix} 0.29 \\ 0.29 \\ 0.29 \\ -0.5 \\ 0 \\ -0.5 \\ 0.29 \\ 0.29 \\ 0.29 \end{bmatrix}.$$

Reducing from $7 \times 9 = 63$ to $2 + 2 \times 7 + 2 \times 9 = 34$ is done by the decomposition

$$W = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T.$$

Exercise 21.3. 1. First we compute these matrices as

$$AA^T = \begin{bmatrix} 2a^2 & 0 \\ 0 & 4a^2 \end{bmatrix}, \qquad A^TA = \begin{bmatrix} a^2 & 0 & a^2 & 0 \\ 0 & 0 & 0 & 0 \\ a^2 & 0 & a^2 & 0 \\ 0 & 0 & 0 & 4a^2 \end{bmatrix}.$$

The eigenvalue / eigenvector pairs of AA^T are evidently $\lambda_1 = 4a^2$ with $\mathbf{u}_1 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $\lambda_2 = 2a^2$ with $\mathbf{u}_2 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. We sort them this way because $4a^2 > 2a^2$. For the SVD we only need the eigenvectors of A^TA corresponding to these two eigenvalues. It is immediate that $\mathbf{v}_1 = [0\ 0\ 0\ 1]^T$ and $\mathbf{v}_2 = [1\ 0\ 1\ 0]^T$, which normalizes to $[1/\sqrt{2}\ 0\ 1/\sqrt{2}\ 0]^T$. Hence the SVD of A is

$$\begin{aligned} A &= 2a \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} + \sqrt{2}a \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2a & 0 & 0 & 0 \\ 0 & \sqrt{2}a & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{bmatrix}}_{V^T}. \end{aligned}$$

2. The dimensions of the four fundamental subspaces are given by the number of rows and columns in the matrices U, Σ, V^T. We get that:

- $\dim(\mathrm{col}(A)) = \mathrm{rank}(A) = 2$
- $\dim(\mathrm{null}(A^T)) = (\text{number of zero rows in } \Sigma) = 0$
- $\dim(\mathrm{row}(A)) = \mathrm{rank}(A) = 2$
- $\dim(\mathrm{null}(A)) = (\text{number of zero columns in } \Sigma) = 2$

Exercise 21.4. 1. There are many examples, one is $A = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{smallmatrix}\right]$. The matrix $AA^T \in \mathcal{M}_{3\times 3}$ is diagonal with $1, 4, 9$ on its diagonal, so those are its eigenvalues. The singular values of A are the positive square roots of these numbers, and those are $1, 2, 3$.

2. Take the left singular vectors to be same as the right ones. Let $\sigma_1 = 4$ (to clear denominators) be the only signular value (because rank is 1). By SVD we get

$$A = \underbrace{\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}}_{V^T} = \begin{bmatrix} 2 & 0 \\ 2\sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}.$$

3. Since A is symmetric, its singular values are its eigenvalues. Since there are many zeros, the eigenvalue / eigenvector pairs can be found by sight:

$$\begin{aligned} \lambda_1 &= \sigma_1 = 2 \\ \lambda_2 &= \sigma_2 = 1 \\ \lambda_3 &= \sigma_3 = 1 \end{aligned} \qquad \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

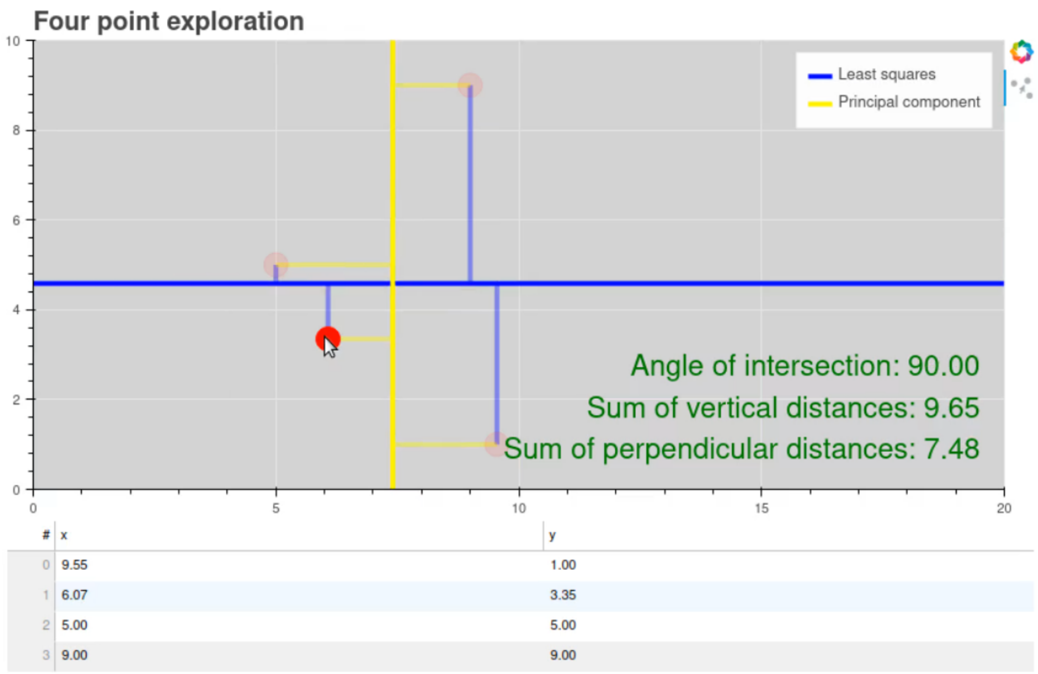
The last eigenvalue is zero because the matrix has two equal rows (so the determinant is 0). The approximations then are:

$$\text{rank } 1: \quad \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T = 2 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

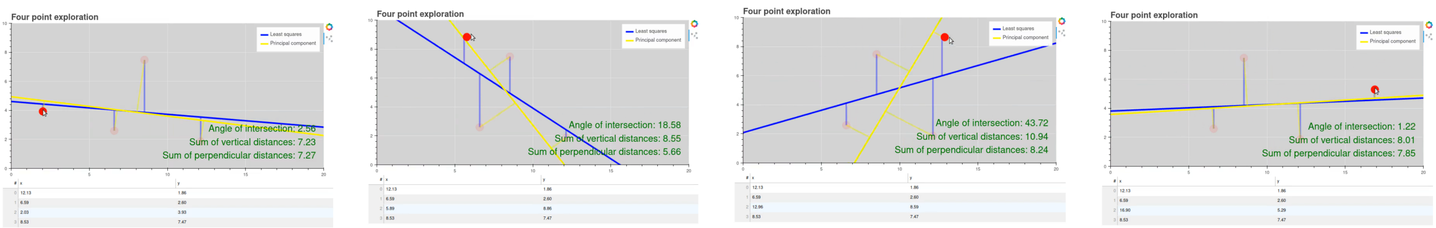
$$\text{rank } 2: \quad \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{other rank } 2: \quad \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_3 \mathbf{u}_3 \mathbf{v}_3^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

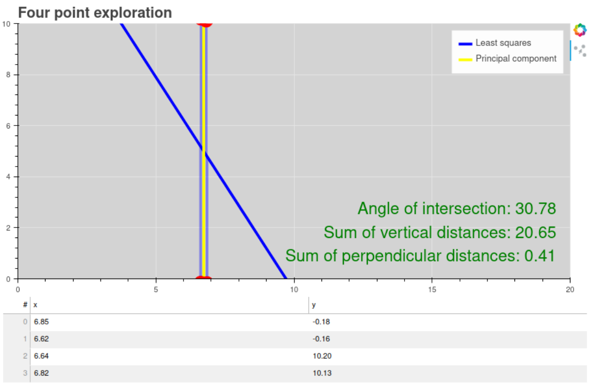
Exercise 22.1. 1. The largest possible angle is $\frac{\pi}{2} = 90°$. There are many arrangements that give this, one of which is described below.



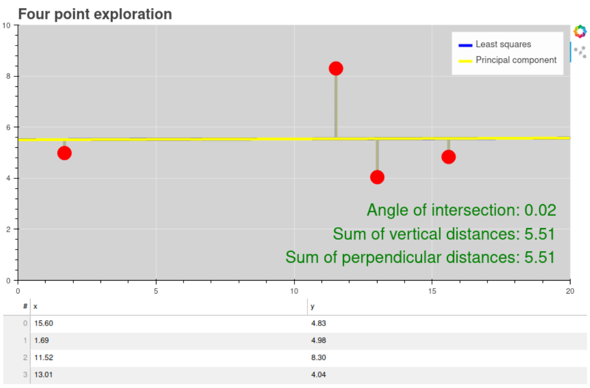
A justification that such an angle is possible comes from "flipping" the yellow line around. At first, the angle is almost 0° with the blue line, and at the end, it is 180° = 0°, and as the movement is continuous, by the intermediate value theorem the angle must have been 90° at some point.



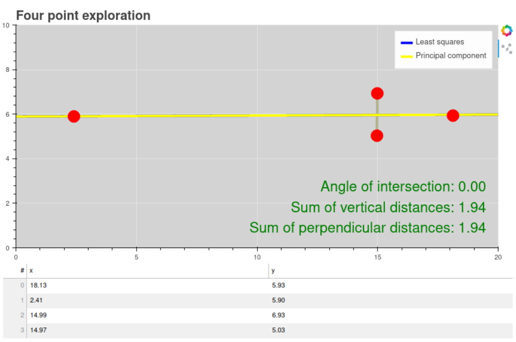
2. The largest possible difference can be created by making the perpendicular distances zero and the vertical distances as large as possible. In the given frame, that maximal difference should be 20 units, since the height of the frame is 10 units. Such an example is given below (due to rendering and approximation errors, the difference is larger than 20).

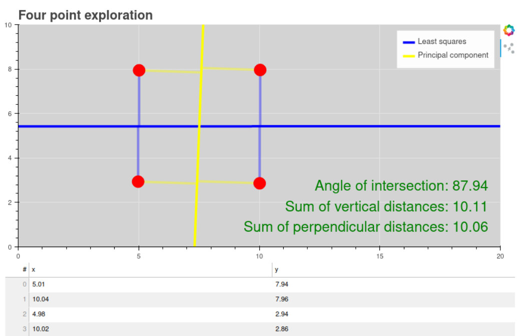


One way to get the same distances is to make sure the least squares and principal component lines are both horizontal. Then the perpendicular distances equal the vertical distances.



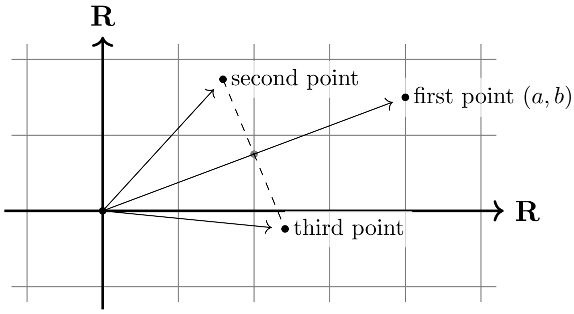
In special cases, this occurs with either a single pair of points can be placed an equal distance away from a horizontal line, or both pairs points can be placed an equal distance away.





In the second case the principal component line is vertical due to rounding errors. Both eigenvalues should be the same.

Exercise 22.3. Since the first principal component solves the perpendicular least squares problem, we choose one point to be exactly $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, and the other two to lie the same distance on either side of this eigenvector. We choose the distance to be $\ell = \sqrt{a^2 + b^2}/4$ so that the two other points do not dominate the first point. The idea is given in the picture below.



We now construct these points explicitly and perform PCA on the data to confirm that the result will be as desired. To find the coordinates of the other two points, note that the slope of the line to (a, b) is $\frac{b}{a}$. So the two other points lie on the line with slope $\frac{-a}{b}$ which goes through $(\frac{a}{2}, \frac{b}{2})$. The equation of the line is given by

$$\frac{-a}{b} = \frac{y - \frac{b}{2}}{x - \frac{a}{2}} \qquad \Longleftrightarrow \qquad f(x) = y = \frac{-a}{b}x + \left(\frac{a^2}{2b} + \frac{b}{2}\right).$$

To find the points a distance ℓ along this line from $(\frac{a}{2}, \frac{b}{2})$, we solve for x in the equality

$$\begin{aligned}
\frac{\sqrt{a^2+b^2}}{4} &= \sqrt{\left(\frac{a}{2} - x\right)^2 + \left(\frac{b}{2} - f(x)\right)^2} \\
&= \sqrt{\left(\frac{a}{2} - x\right)^2 + \left(\frac{b}{2} - \left(\frac{-a}{b}x + \frac{a^2}{2b} + \frac{b}{2}\right)\right)^2} \\
&= \sqrt{\left(\frac{a}{2} - x\right)^2 + \left(\frac{a}{b}x - \frac{a^2}{2b}\right)^2} \\
&= \sqrt{\left(\frac{a}{2} - x\right)^2 + \frac{a^2}{b^2}\left(x - \frac{a}{2}\right)^2} \\
&= \sqrt{\left(\frac{a}{2} - x\right)^2 \left(1 + \frac{a^2}{b^2}\right)}.
\end{aligned}$$

This simplifies to $x = \frac{2a \pm b}{4}$, so the data we have is

$$A = \begin{bmatrix} a & \frac{2a-b}{4} & \frac{2a+b}{4} \\ b & f(\frac{2a-b}{4}) & f(\frac{2a+b}{4}) \end{bmatrix} = \begin{bmatrix} a & \frac{2a-b}{4} & \frac{2a+b}{4} \\ b & \frac{2b+a}{4} & \frac{2b-a}{4} \end{bmatrix}.$$

For PCA, we need to mean-center the data first. The mean of x-coordinates is $2a/3$ and the mean of the y-coordinates is $2b/3$, so after subtracting $2a/3$ from the first row and $2b/3$ from the second row, we get the mean centered data to be

$$M = \begin{bmatrix} \frac{a}{3} & \frac{-2a-3b}{12} & \frac{3b-2a}{12} \\ \frac{b}{3} & \frac{3a-2b}{12} & \frac{-3a-2b}{12} \end{bmatrix} \qquad \Longrightarrow \qquad S = \frac{MM^T}{2} = \begin{bmatrix} \frac{4a^2+3b^2}{48} & \frac{ab}{48} \\ \frac{ab}{48} & \frac{3a^2+4b^2}{48} \end{bmatrix}.$$

With the help of a computer, we find the eigenvalues and eigenvectors of this symmetric matrix to be

$$\lambda_1 = \frac{a^2+b^2}{12}, \quad \mathbf{u}_1 = \begin{bmatrix} a/b \\ 1 \end{bmatrix}, \qquad \lambda_2 = \frac{a^2+b^2}{16}, \quad \mathbf{u}_2 = \begin{bmatrix} -b/a \\ 1 \end{bmatrix}.$$

It looks like we are done, but the eigenvector $\left[\begin{smallmatrix} a/b \\ 1 \end{smallmatrix}\right]$ is for the mean-centered data, so we need to shift it back. Hence the first eigenvector for the original data is

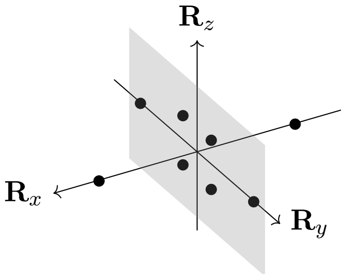
$$\begin{bmatrix} a/b \\ 1 \end{bmatrix} + \begin{bmatrix} 2a/3 \\ 2b/3 \end{bmatrix} = \begin{bmatrix} \frac{2ab+3a}{3b} \\ \frac{2b+3}{3} \end{bmatrix} = \frac{2b+3}{3b} \begin{bmatrix} a \\ b \end{bmatrix},$$

which is indeed a multiple of $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, as desired.

Exercise 22.4. 1. We folow a similar method as in Question 1, placing what were the second and third points in the second eigenvector direction. We make some other changes:

- Since we need at least 4 points, but cannot place three points in a line, we split up what were the second and third points.
- To ensure the second principal component is $\left[\begin{smallmatrix} 0 & 1 & 0 \end{smallmatrix}\right]^T$, we place further points along the second principal component axis.
- To ensure that the data is mean-centered, we mirror all the points.

This construction is demonstrated in the picture below left (the plane $x = 0$ is emphasized in gray), with the points in the matrix below right.



$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & -1 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 & 0 \end{bmatrix}$$

It is evident that no three points lie on a line. The mean-centered matrix M is the same as A, since the mean of each row is 0. The sample covariance matrix S and its eigenvectors are

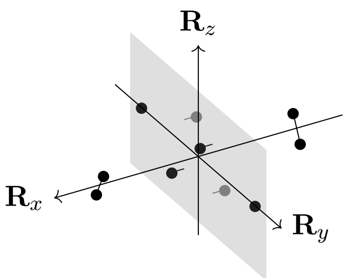
$$S = \begin{bmatrix} \frac{2}{7} & 0 & 0 \\ 0 & \frac{9}{28} & 0 \\ 0 & 0 & \frac{1}{28} \end{bmatrix}, \qquad \begin{aligned} \mathbf{u}_1 &= \left[\begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix}\right]^T \\ \mathbf{u}_2 &= \left[\begin{smallmatrix} 0 & 1 & 0 \end{smallmatrix}\right]^T \\ \mathbf{u}_3 &= \left[\begin{smallmatrix} 0 & 0 & 1 \end{smallmatrix}\right]^T \end{aligned}.$$

Hence the presented data satisfies the given conditions.

2. Columns 1,2,7,8 lie in plane $z = 0$ and columns 3-8 lie in plane $x = 0$ (emphasized in the picture above). To fix theses issues, we take two steps:

- For the first issue, we split the points $(1, 0, 0)$ and $(-1, 0, 0)$ into two points just above and below the x-axis. We shift them in equal but opposite diections along the y-axis so that the new points are not on a plane.
- For the the second issue, we move the four points in columns 3-6 by equal but opposite distances in the x-direction.
- To ensure the two solutions do not conflict, the shifting magnitudes are different.

The new data is given below left (with lines indicating shifts from the previous data), and the new matrix is given below right.



$$A = \begin{bmatrix} 1 & 1 & -1 & -1 & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & -1 \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 & 0 \end{bmatrix}$$

By sight we confirm that no four points of these samples line in the same plane. The data is still mean-centered (since we added equal but opposite values to each row), and the sample covariance matrix with its eigenvectors is

$$S = \begin{bmatrix} \frac{65}{144} & 0 & 0 \\ 0 & \frac{145}{576} & 0 \\ 0 & 0 & \frac{5}{144} \end{bmatrix}, \qquad \begin{aligned} \mathbf{u}_1 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \\ \mathbf{u}_2 &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \\ \mathbf{u}_3 &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \end{aligned}.$$

Hence all the conditions are satisfied.

Exercise 23.1. 1. T_1 is linear, and its matrix is a permutation matrix:

$$T_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

T_2 is not linear, as $T_2 \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2e^2 \\ 0 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 4e \\ 0 \end{smallmatrix}\right] = 2T_2 \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

T_3 is not linear, as $T_3 \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 9 \\ 0 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right] = 3T_3 \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$.

T_4 is not linear, as $T_4 \left[\begin{smallmatrix} \sqrt{\pi} \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \sqrt{2} T_4 \left[\begin{smallmatrix} \sqrt{\pi/2} \\ 0 \end{smallmatrix}\right]$.

T_5 is not linear, as $T_5 \left[\begin{smallmatrix} 0 \\ 0 \\ 3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \\ 9 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right] = 3T_5 \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$.

T_6 is linear, and its matrix is the zero matrix:

$$T_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

T_7 is linear, and its matrix can be found by what it does to each variable:

$$T_7 = \begin{bmatrix} -3 & 0 & 0 \\ -0 & 1 & 1 \end{bmatrix}.$$

T_8 is linear, and its matrix can be found by what it does to each variable:

$$T_8 = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

2. The three conditions that are given can be simplified using the following observations:

$$T_5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad T_8 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad T_8 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Using this, we get a clearer description of what S does to $\mathbf{R}^3$:

$$S \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad S \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad S \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

To get the matrix of S, we first describe what S does on the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Note that

$$S \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = S \left(\frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \frac{1}{2} S \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} S \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

for $\mathbf{e}_1$, and

$$S \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = S \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = S \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} S \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} S \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ 1/2 \end{bmatrix}$$

for $\mathbf{e}_3$. For $\mathbf{e}_2$ we already know what happens. Applying the proof of Theorem 18.9 on the construction of the matrix associated to a linear transformation, we get that the matrix of S is

$$S = \begin{bmatrix} -1/2 & 1 & 3/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}.$$

Exercise 24.1. We place z in the denominator and multiply by the conjugate:

$$\frac{1}{z} = \frac{1}{x+yi} = \frac{1}{x+yi} \cdot \frac{x-yi}{x-yi} = \frac{x-yi}{x^2+y^2} = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2} i.$$

We are allowed to multiply by the conjugate, since at least one of x, y are non-zero. The answer makes sense also because at least one of x, y are non-zero. That is, we are never dividing by zero.

Exercise 24.2. 1.

$$\begin{aligned} \overline{z+w} &= \overline{(x+yi)+(a+bi)} \\ &= \overline{(x+a)+(y+b)i} \\ &= (x+a)-(y+b)i \\ &= (a-yi)+(a-bi) \\ &= \overline{z}+\overline{w} \end{aligned}$$

2.

$$\begin{aligned} \overline{zw} &= \overline{(x+yi)(a+bi)} \\ &= \overline{xa+xbi+yai-yb} \\ &= \overline{(xa-yb)+(xb+ya)i} \\ &= (xa-yb)-(xb+ya)i \\ &= xa-yb-xbi-yai \\ &= (x-yi)a-(x-yi)bi \\ &= (x-yi)(a-bi) \\ &= \overline{z}\;\overline{w} \end{aligned}$$

3.

$$\overline{\overline{z}} = \overline{\overline{x+yi}} = \overline{x-yi} = x+yi = z$$

4.

$$z+\overline{z} = (x+yi)+(x-yi) = (x+x)+(y-y)i = 2x$$

5.

$$z-\overline{z} = (x+yi)-(x-yi) = (x-x)+(y+y)i = 2yi$$

6. Since $zz^{-1} = 1$, we have that
$$z^{-1} = \frac{1}{z} = \frac{1}{x+yi} = \frac{1}{x+yi}\frac{x-yi}{x-yi} = \frac{x-yi}{x^2+y^2} = \frac{\overline{z}}{|z|^2}.$$

7. Suppose that $|z| = 0$. Then
$$0 = |z| = \sqrt{x^2+y^2} \quad \Longrightarrow \quad 0 = x^2+y^2.$$
Since $x^2 \geqslant 0$ and $y^2 \geqslant 0$, but their sum is equal to zero, it must be that $x = y = 0$, so $z = 0$. Conversely, suppose that $z = 0$. Then $|z| = \sqrt{0^2} = 0$.

8. $$|\overline{z}| = |\overline{x+yi}| = |x-yi| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |x+yi| = |z|$$

9. $$\begin{aligned} |zw| &= |(x+yi)(a+bi)| \\ &= |xa+xbi+yai-yb| \\ &= |(xa-yb)+(xb+ya)i| \\ &= \sqrt{(xa-yb)^2+(xb+ya)^2} \\ &= \sqrt{(xa)^2-2xayb+(yb)^2+(xb)^2+2xbya+(ya)^2} \\ &= \sqrt{(xa)^2+(yb)^2+(xb)^2+(ya)^2} \\ &= \sqrt{(x^2+y^2)(a^2+b^2)} \\ &= \sqrt{x^2+y^2}\sqrt{a^2+b^2} \\ &= |z||w| \end{aligned}$$

10. For this question we work backwards, doing invertible operations (adding / subtracting, multipliying / dividing by nonzero numbers):

$$\begin{aligned}
& & |z+w| &\leqslant |z|+|w| & & \\
&\Longleftrightarrow & |(x+yi)+(a+bi)| &\leqslant |x+yi|+|a+bi| & &\text{(expanding)} \\
&\Longleftrightarrow & |(x+a)+(y+b)i| &\leqslant |x+yi|+|a+bi| & &\text{(expanding)} \\
&\Longleftrightarrow & \sqrt{(x+a)^2+(y+b)^2} &\leqslant \sqrt{x^2+y^2}+\sqrt{a^2+b^2} & &\text{(definition)} \\
&\Longleftrightarrow & (x+a)^2+(y+b)^2 &\leqslant x^2+y^2+2\sqrt{(x^2+y^2)(a^2+b^2)}+a^2+b^2 & &\text{(squaring)} \\
&\Longleftrightarrow & x^2+2ax+a^2+y^2+2yb+b^2 &\leqslant x^2+y^2+2\sqrt{(x^2+y^2)(a^2+b^2)}+a^2+b^2 & &\text{(expanding)} \\
&\Longleftrightarrow & 2ax+2yb &\leqslant 2\sqrt{(x^2+y^2)(a^2+b^2)} & &\text{(cancelling)} \\
&\Longleftrightarrow & ax+yb &\leqslant \sqrt{(x^2+y^2)(a^2+b^2)} & &\text{(dividing by 2)} \\
&\Longleftrightarrow & (ax)^2+2axyb+(yb)^2 &\leqslant x^2a^2+x^2b^2+y^2a^2+y^2b^2 & &\text{(squaring)} \\
&\Longleftrightarrow & 2axyb &\leqslant x^2b^2+y^2a^2 & &\text{(cancelling)} \\
&\Longleftrightarrow & 0 &\leqslant x^2b^2-2axyb+y^2a^2 & &\text{(rearranging)} \\
&\Longleftrightarrow & 0 &\leqslant (xb-ya)^2 & &\text{(rearranging)}
\end{aligned}$$

This last line is clearly a true statement, and since all operations were reversible, the first line is also true.

Exercise 25.2. 1. First we note that, for every $n \in \mathbf{N}$,
$$\|\sin(nx)\|^2 = \|\cos(nx)\|^2 = \pi,$$

which follows by double angle identities and subsitution. Then we compute:

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f^2(x)\ dx = \frac{2\pi}{2\pi} = 1\\
a_1 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(x)\ dx = -\frac{2}{\pi}\int_0^{\pi} \sin(x)\ dx = -\frac{4}{\pi}\\
a_2 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(2x)\ dx = \frac{2}{\pi}\int_0^{\pi} \sin(2x)\ dx = 0\\
a_3 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(3x)\ dx = -\frac{2}{\pi}\int_0^{\pi} \sin(3x)\ dx = -\frac{4}{3\pi}\\
a_4 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(4x)\ dx = \frac{2}{\pi}\int_0^{\pi} \sin(4x)\ dx = 0\\
a_3 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(3x)\ dx = -\frac{2}{\pi}\int_0^{\pi} \sin(5x)\ dx = -\frac{4}{5\pi}
\end{aligned}$$

This follows by the periodicity of sin. For cos we have even simpler results, as the integral from 0 to π of $\cos(nx)$ is already zero. That is,

$$b_1 = b_2 = b_3 = b_4 = b_5 = 0.$$

2. For $n = 4$, the samples are $(0, -1)$, $(\pi/2, -1)$, $(\pi, 1)$, $(3\pi/2, 1)$, and we get the Fourier transform to be

$$\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -i & -1 & i\\ 1 & -1 & 1 & -1\\ 1 & i & -1 & -i\end{bmatrix}\begin{bmatrix}-1\\ -1\\ 1\\ 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & i-1 & 0 & -i-1\end{bmatrix}.$$

3.

Index of notation

Index

Key ideas

Secondary ideas

Algorithms

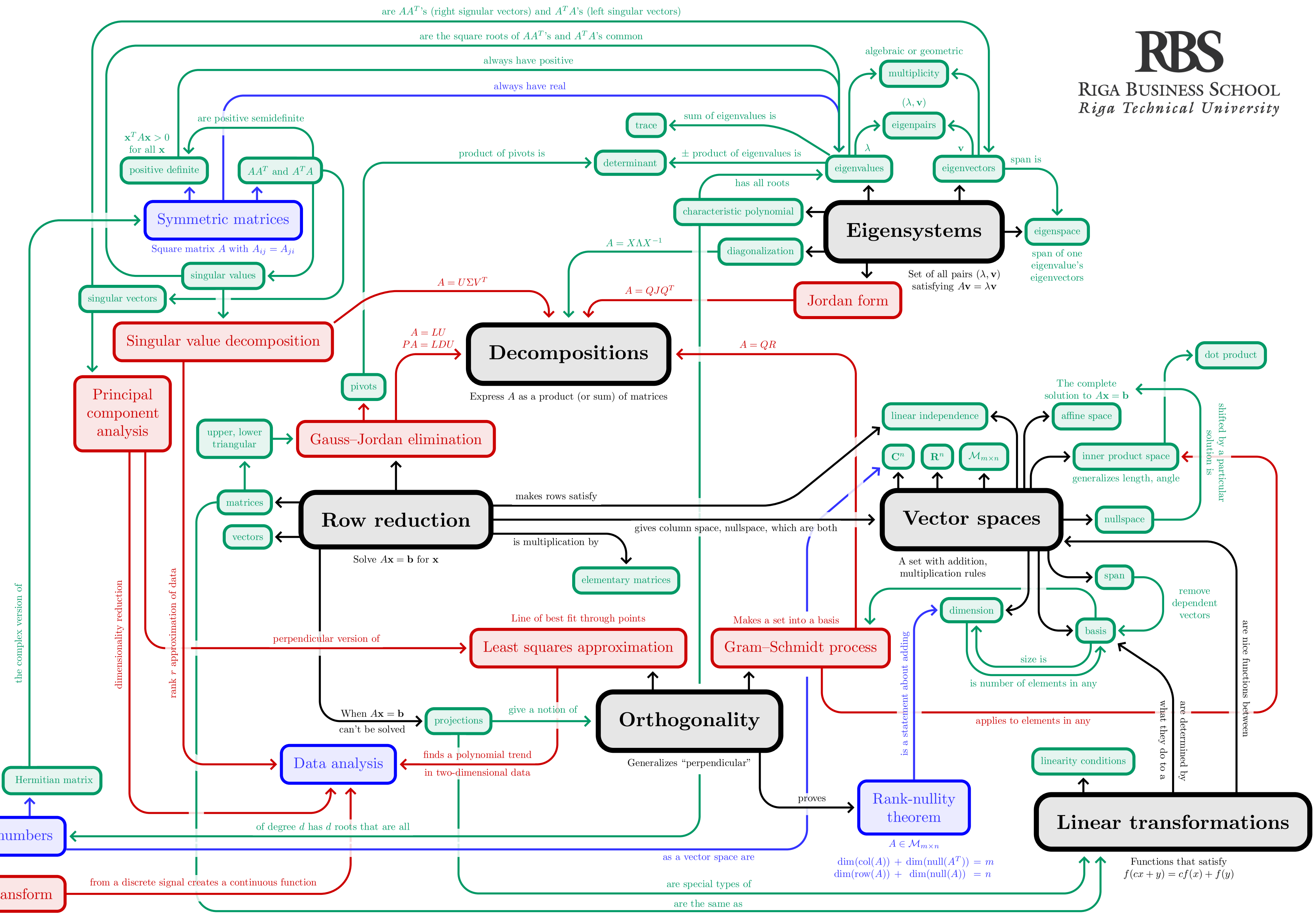
Terms and concepts

Matrices

- A, B, C : any matrix
- D : diagonal
- F : Fourier
- J : Jordan form
- L : lower triangular
- P : permutation
- P : projection
- Q : orthonormal columns
- S : symmetric
- U : upper triangular
- U : right singular vectors as columns
- V : left singular vectors as columns
- X : (generalized) eigenvectors as columns
- Λ : diagonal with eigenvalues on diagonal
- Σ : diagonal with singular values on diagonal

Numbers

- i : imaginary number
- r : rank
- z : complex number
- λ : eigenvalue
- σ : singular value



Linear Algebra

as taught in the RTU Riga Business School BITL program

2022. Jānis Lazovskis