Assignment 5

Introduction to Linear Algebra

Material from Lectures 18 - 24
Due Wednesday, April 13, 2022

18.4 Let A, B, C be any 3×3 matrices, with C digonalizable.

(a) Show that $\text{trace}(AB) = \text{trace}(BA)$.

(b) Use that above to show that $\text{trace}(C)$ is the sum of the three eigenvalues of C. *Hint: Split up the diagonalization of C into two matrices.*

(c) Suppose that the eigenvalues of C are $1, \frac{1}{2}, \frac{1}{3}$. Show why the limit $\lim\limits_{n\to\infty} C^n$ exists, and why it has rank 1.

20.3 Let $J \in \mathcal{M}_{6\times 6}$ be a matrix in Jordan form with two eigenvalues 3 (having algebraic multiplicity 4 and geometric multiplicity 2) and -3 (having algebraic multiplicity 2 and geometric multiplicity 1).

(a) How many Jordan blocks will J have? Give the two possibilities for their sizes.

(b) Suppose that the Jordan blocks of J all have the same size. Find a matrix B that is similar to J and has no zero entries.

(c) For the matrix B from part (2.), find all its generalized eigenvectors.

21.4 (a) Construct a 3×4 matrix with singular values $1, 2, 3$.

(b) Construct a 2×2 rank 1 matrix with right singular vectors $\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}, \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$.

(c) Find the rank 1 and rank 2 approximations for

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hint: Since two eigenvalues are the same, there are two rank 2 approximations!

22.1 This question is about the 4 point interactive found on the course website (link here).

(a) Create an arrangement of the points with the largest angle possible between the two approximations that you can find. Do you think any angle is possible? Justify your reasoning.

(b) Create an arrangement of the points with the largest difference between the sums of the distances that you can find. Besides all points being on a line, what situations give the same sums of distances?

Submit an image on ORTUS for each part.

23.1 For this question, the vector $T_i(\mathbf{x})$ is simply written $T_i\mathbf{x}$, to both ease notation and as a reminder that linear transformatons are simply matrices. You are given the following transformations T_i:

$$T_1 \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} w \\ y \\ z \\ x \end{bmatrix} \qquad T_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2e^y \\ x \end{bmatrix} \quad T_3 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \qquad T_4 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sin(x^2+y^2) \\ \cos(x^2+y^2) \end{bmatrix}$$

$$T_5 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y+x \\ 0 \\ x^2-y \end{bmatrix} \quad T_6 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad T_7 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3x \\ z+y \end{bmatrix} \quad T_8 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+2y \\ y+z \\ 0 \end{bmatrix}$$

(a) Which of the T_i are linear? For those that are not, give a counterexample in which one of the linearity conditions fail.

(b) Let $S\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation for which

$$ST_5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad ST_8 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad ST_8 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Construct the 3×3 matrix of S.

24.1 Show that every complex number $z = x + iy$ for which at least one of x and y are not zero has an inverse. That is, find $w \in \mathbf{C}$ for which $zw = 1$.