Assignment 4

Introduction to Linear Algebra

Material from Lectures 12 - 17

Due Monday, March 14, 2022

Exercises are taken from the lecture notes.

12.3 Consider the 2-dimensional subspace $H \subseteq \mathbf{R}^4$ defined by

$$H = \left\{ (x,y,z,w) \in \mathbf{R}^4 \;:\; \begin{array}{rcl} 2x+3y-w & = & 0, \\ y-z+2w & = & 0. \end{array} \right\}$$

(1.) Express H as a span of two vectors.

(2.) Apply the Gram–Schmidt process to the two vectors from above to get H as a span of two orthonormal vectors.

(3.) The space $\mathbf{R}^4$ has the xy-plane as a 2-dimensional subspace, with basis $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Give the change of basis matrix from the two vectors in part 2. to these two vectors.

13.4 Let $P(\mathbf{R})$ be the vector space of all polynomials $\mathbf{R} \to \mathbf{R}$, with scalar multiplication and polynomial addition defined as you would expect. You may assume that the following is an inner product on $P(\mathbf{R})$:

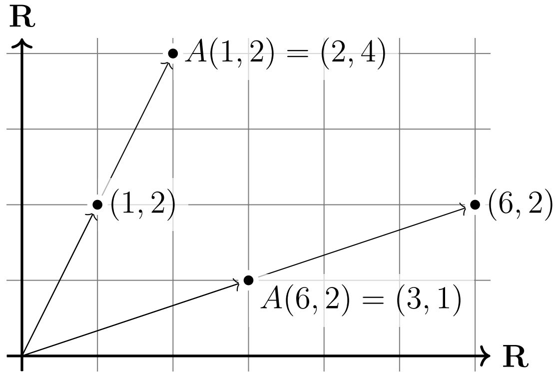
$$\langle p(x), q(x) \rangle = \int_0^\infty p(x)q(x)e^{-x}\,dx.$$

(1.) Check that $p(x) = 2x-1$ and $q(x) = x+3$ are not orthogonal to each other.

(2.) Using the Gram–Schmidt process on $p(x)$ and $q(x)$ as in part 1., find a polynomial $r(x) \in P(\mathbf{R})$ that is orthogonal to $p(x)$. Give your answer as $r(x) = ax+b$, for $a, b \in \mathbf{Z}$.

14.3 Let $A \in \mathcal{M}_{n\times n}$. Show that $\det(A) = 0$ is equivalent to saying that there is a nonzero vector $\mathbf{x}$ for which $A\mathbf{x} = 0$.

16.5 Let $A\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the 2×2 matrix for which $A\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix}6\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$. This is described in the picture below.



(1.) What is the eigensystem of A? Express $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ as linear combinations of the eigenvectors of A.

(2.) Using the task above, compute $A\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $A[\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Use this to construct the matrix of A.

(3.) Using eigenvalues, explain why A is invertible.

17.1 Let $A \in \mathcal{M}_{n\times n}$ and let $\chi(t)$ be its charactristic polynomial.

(1.) Show that $\chi(0) = (-1)^n \det(A)$. That is, show that the constant term in $\chi(t)$ is $(-1)^n$ times the determinant of A.

(2.) Show that the coefficient of t^{n-1} in $\chi(t)$ is $-\text{trace}(A)$.