Midterm

Introduction to Linear Algebra

Material from Lectures 1 - 12

Fall 2021

- This midterm lasts 1 hour and 40 minutes.
- This midterm has 8 questions. Each question is worth 5 points.
- Your grade will be Q1+Q2+(highest 5 from Q3 - Q8). That is, the lowest scoring question from Q3 - Q8 will be dropped.
- This is an open-book midterm. All work submitted must be your own. You may not communicate with other students during the midterm.
- Write your answer for each question on a separate page. Do not answer more than one question on a single page.
- Questions 1 and 2 do not need justification. Questions 3 - 8 require justification.

Question	Grade
1	
2	
3	
4	
5	
6	
7	
8	
Total	/35

1. Answer the following True / False questions. You do not need to show your reasoning.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, then $|\mathbf{u} \cdot \mathbf{v}| \leqslant 1$

(b) The matrix $\left[\begin{smallmatrix} 3 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is upper triangular.

(c) Any two vectors in $\mathbf{R}^n$ are either parallel or perpendicular.

(d) Any two vectors in a basis of $\mathbf{R}^n$ are either parallel or perpendicular.

(e) Any two vectors in an orthogonal basis of $\mathbf{R}^n$ are either parallel or perpendicular.

(f) If all vectors in a set are orthogonal to each other, then they are linearly independent.

(g) The set of 3×3 matrices that are not symmetric is a vector subspace of $\mathcal{M}_{3\times 3}$.

(h) The function $\langle f, g \rangle = \displaystyle\int_0^1 (f(x) + g(x))^2 \, dx$ is an inner product on $C[0, 1]$.

(i) Any symmetric matrix with non-negative entries and a zero diagonal is a distance matrix for some appropriate inner product space.

(j) The determinant of a rank one matrix is always 0.

2. Answer the following short answer questions. You do not need to show your work.

(a) If $\mathbf{v} \in \mathbf{R}^n$, then the dimensions of $(\mathbf{v}^T\mathbf{v})(\mathbf{v}\mathbf{v}^T)$ are ______

(b) The length of the vector $\begin{bmatrix} a \\ b \\ a-b \end{bmatrix}$ is ______

(c) The product abc in the LU-decomposition

$$\underbrace{\begin{bmatrix} 6 & 0 & -2 \\ 1 & 3 & 4 \\ -3 & -8 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 6 & 0 & -2 \\ 0 & 3 & 13/3 \\ 0 & 0 & 113/9 \end{bmatrix}}_{U}$$

is ______

(d) If the rank of $A \in \mathcal{M}_{5\times 6}$ is 3, then the rank of A^T is ______

(e) The determinant of the matrix $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}$ is $\det(A) =$ ______

3. Consider the following vectors and matrices.

$$\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ -\sqrt{6} \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & \sqrt{6} & 2 \\ 3 & 0 & 0 \\ \sqrt{6} & 1 & 7/4 \end{bmatrix}$$

(a) Find the angle between $\mathbf{v}$ and $\mathbf{w}$.

(b) Find two different triples $a, b, c \in \mathbf{R}$ so that $a\mathbf{v} + b\,\mathrm{diag}(\mathbf{w}\mathbf{w}^T) + cA\mathbf{w} = \left[\begin{smallmatrix} 2 \\ -1 \\ -5/4 \end{smallmatrix}\right]$.

Recall that $\mathrm{diag}(M) = \begin{bmatrix} M_{11} \\ M_{22} \\ \vdots \\ M_{nn} \end{bmatrix}$ is the column vector of the diagonal entries of M.

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4. Consider the matrix $A = \begin{bmatrix} a & b & c & d & e \\ a & 0 & c & 0 & e \\ 0 & b & 0 & d & 0 \end{bmatrix}$, for $a, b, c, d, e \in \mathbf{R}_{\neq 0}$.

(a) Express $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ as a linear combination of the basis vectors from the column space and left nullspace of A.

(b) Express $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ as a linear combination of the basis vectors from the row space and nullspace of A.

You may place restrictions on a, b, c, d, e to avoid division by 0.

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5. Let $S \subseteq \mathcal{M}_{2\times 2}$ be the space of symmetric 2×2 matrices.

(a) Show that $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for S.

(b) Extend B to an orthogonal basis for $\mathcal{M}_{2\times 2}$.

You may use the fact that $\mathcal{M}_{2\times 2} = \text{span}(B')$, where

$$B' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

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6. Consider the hyperplane $H = \{(x, y, z, w) \ : \ 2x - 4y + 2z - 1w = 0\} \subseteq \mathbf{R}^4$.

   (a) Give a basis for this subspace.

   (b) Find two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{R}^4$ not in H, for which:

   - the projection of $\mathbf{a}$ onto H is not the same as the projection of $\mathbf{b}$ onto H, and
   - the error in projecting both $\mathbf{a}$ and $\mathbf{b}$ onto H is 5.

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7. Consider the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 3 & 2 & 3 \end{bmatrix}$.

(a) Give the LU-decomposition of A.

(b) Give the 2nd row of R in the QR-decomposition of A.

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8. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$.

(a) How many nonzero terms are in the permutation formula of $\det(A)$?

(b) Express (do not evaluate) the determinant of A using the recursive formula.

(c) Use the pivot definition of the determinant to evaluate your answer from part (b).
