Midterm

Introduction to Linear Algebra

Material from Lectures 1 - 12

Fall 2021

- This midterm lasts 1 hour and 40 minutes.
- This midterm has 8 questions. Each question is worth 5 points.
- Your grade will be Q1+Q2+(highest 5 from Q3 - Q8). That is, the lowest scoring question from Q3 - Q8 will be dropped.
- This is an open-book midterm. All work submitted must be your own. You may not communicate with other students during the midterm.
- Write your answer for each question on a separate page. Do not answer more than one question on a single page.
- Questions 1 and 2 do not need justification. Questions 3 - 8 require justification.

Question	Grade
1	
2	
3	
4	
5	
6	
7	
8	
Total	/35

1. Answer the following True / False questions. You do not need to show your reasoning.

   (a) If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, then $|\mathbf{u} \cdot \mathbf{v}| \leqslant 1$
   True. This is the Cauchy–Schwarz inequality.

   (b) The matrix $\left[\begin{smallmatrix} 3 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is upper triangular.
   True. Upper triangular means everyhing below the diagonal is 0.

   (c) Any two vectors in $\mathbf{R}^n$ are either parallel or perpendicular.
   False. A counterexample is $(1, 0)$ and $(1, 1)$, which are neither.

   (d) Any two vectors in a basis of $\mathbf{R}^n$ are either parallel or perpendicular.
   False. A counterexample is $\{(1, 0), (1, 1)\}$ as a basis of $\mathbf{R}^2$.

   (e) Any two vectors in an orthogonal basis of $\mathbf{R}^n$ are either parallel or perpendicular.
   True. Orthogonal means perpendicular.

   (f) If all vectors in a set are orthogonal to each other, then they are linearly independent.
   False. A couterexample is the set $\{(0, 0), (1, 0)\} \subseteq \mathbf{R}^2$. True is also acepted, because False only happens with the zero vector, and with the statement as given, it is reasonable to assume "non-zero vectors" was implied.

   (g) The set of 3×3 matrices that are not symmetric is a vector subspace of $\mathcal{M}_{3\times3}$.
   False. The zero matrix is symmetric, and must be in every vector subspace of $\mathcal{M}_{3\times3}$.

   (h) The function $\langle f, g\rangle = \displaystyle\int_0^1 (f(x) + g(x))^2 \, dx$ is an inner product on $C[0, 1]$.
   False. It is not bilinear nor multiplicative.

   (i) Any symmetric matrix with non-negative entries and a zero diagonal is a distance matrix for some appropriate inner product space.
   False. A counterexample is given in Example 10.9 in the lecture notes.

   (j) The determinant of a rank one matrix is always 0.
   False. A counterexample is $[1]$, which is a 1×1 rank one matrix with $\det([1]) = 1$.

2. Answer the following short answer questions. You do not need to show your work.

(a) If $\mathbf{v} \in \mathbf{R}^n$, then the dimensions of $(\mathbf{v}^T\mathbf{v})(\mathbf{v}\mathbf{v}^T)$ are ______ $n \times n$ ______

(b) The length of the vector $\begin{bmatrix} a \\ b \\ a-b \end{bmatrix}$ is ______ $\sqrt{a^2+b^2+(a-b)^2}$ ______

(c) The product abc in the LU-decomposition

$$\underbrace{\begin{bmatrix} 6 & 0 & -2 \\ 1 & 3 & 4 \\ -3 & -8 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 6 & 0 & -2 \\ 0 & 3 & 13/3 \\ 0 & 0 & 113/9 \end{bmatrix}}_{U}$$

is ______ $(1/6)(-1/2)(-8/3) = 2/9$ ______

(d) If the rank of $A \in \mathcal{M}_{5\times 6}$ is 3, then the rank of A^T is ______ 3 ______

(e) The determinant of the matrix $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}$ is $\det(A) =$ ______ -42 ______

3. Consider the following vectors and matrices.

$$\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ -\sqrt{6} \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & \sqrt{6} & 2 \\ 3 & 0 & 0 \\ \sqrt{6} & 1 & 7/4 \end{bmatrix}$$

(a) Find the angle between $\mathbf{v}$ and $\mathbf{w}$.

(b) Find two different triples $a, b, c \in \mathbf{R}$ so that $a\mathbf{v} + b\,\mathrm{diag}(\mathbf{w}\mathbf{w}^T) + cA\mathbf{w} = \left[\begin{smallmatrix} 2 \\ -1 \\ -5/4 \end{smallmatrix}\right]$.

Recall that $\mathrm{diag}(M) = \begin{bmatrix} M_{11} \\ M_{22} \\ \vdots \\ M_{nn} \end{bmatrix}$ is the column vector of the diagonal entries of M.

(a) (2 points) The angle is computed by the formula

$$\cos(\theta) = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{1 + 2\sqrt{6} - 1}{\sqrt{1+4+1}\sqrt{1+6+1}} = \frac{2\sqrt{6}}{\sqrt{6}\sqrt{8}} = \frac{2}{\sqrt{8}} = \frac{1}{\sqrt{2}}.$$

Hence the angle is either $\frac{\pi}{4}$ or $\frac{7\pi}{4}$.

(b) (3 points) First we compute the given vectors:

$$\mathrm{diag}(\mathbf{w}\mathbf{w}^T) = \mathrm{diag}\left(\begin{bmatrix} 1 \\ -\sqrt{6} \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -\sqrt{6} & 1 \end{bmatrix}\right) = \mathrm{diag}\left(\begin{bmatrix} 1 & -\sqrt{6} & 1 \\ -\sqrt{6} & 6 & -\sqrt{6} \\ 1 & -\sqrt{6} & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$$

$$A\mathbf{w} = \begin{bmatrix} 2 & \sqrt{6} & 2 \\ 3 & 0 & 0 \\ \sqrt{6} & 1 & 7/4 \end{bmatrix}\begin{bmatrix} 1 \\ -\sqrt{6} \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 7/4 \end{bmatrix}$$

The given equation can be written as a matrix equation:

$$a\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} + c\begin{bmatrix} -2 \\ 3 \\ 7/4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5/4 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 & -2 \\ -2 & 6 & 3 \\ -1 & 1 & 7/4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5/4 \end{bmatrix}.$$

The augmented matrix takes three row operations to clear below the pivots:

$$\begin{bmatrix} 1 & 1 & -2 & 2 \\ -2 & 6 & 3 & -1 \\ -1 & 1 & 7/4 & -5/4 \end{bmatrix} \xrightarrow{\textit{1st pivot}} \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 8 & -1 & 3 \\ 0 & 2 & -1/4 & 3/4 \end{bmatrix} \xrightarrow{\textit{2nd pivot}} \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 8 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence c is the free variable, and we have $8b - c = 3$ and $a + b - 2c = 2$ as the linear system. Choosing $c = 5$ and $c = -11$ (to make the answers whole numbers) we solve for either $a = 11, b = 1$ and $a = -19, b = -1$, respectively. That is, two (of many) solutions are

$$(a, b, c) = (11, 1, 5) \qquad \text{and} \qquad (a, b, c) = (-19, 1, -11).$$

4. Consider the matrix $A = \begin{bmatrix} a & b & c & d & e \\ a & 0 & c & 0 & e \\ 0 & b & 0 & d & 0 \end{bmatrix}$, for $a, b, c, d, e \in \mathbf{R}_{\neq 0}$.

(a) Express $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ as a linear combination of the basis vectors from the column space and left nullspace of A.

(b) Express $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ as a linear combination of the basis vectors from the row space and nullspace of A.

You may place restrictions on a, b, c, d, e to avoid division by 0.

(a) First we row reduce A:

$$\begin{bmatrix} a & b & c & d & e \\ a & 0 & c & 0 & e \\ 0 & b & 0 & d & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} a & b & c & d & e \\ 0 & -b & 0 & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & c/a & 0 & e/a \\ 0 & 1 & 0 & d/b & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are only two pivots, in columns 1 and 2, so $\operatorname{col}(A) = \operatorname{span}\left\{ \left[\begin{smallmatrix} a \\ a \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} b \\ 0 \\ b \end{smallmatrix}\right] \right\}$. Next we row reduce A^T:

$$\begin{bmatrix} a & a & 0 \\ b & 0 & b \\ c & c & 0 \\ d & 0 & d \\ e & e & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} a & a & 0 \\ 0 & -b & b \\ 0 & 0 & 0 \\ 0 & -d & d \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} a & a & 0 \\ 0 & -b & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is 1 free column, so $\operatorname{null}(A^T) = \operatorname{span}\left\{ \left[\begin{smallmatrix} -1 \\ 1 \\ 1 \end{smallmatrix}\right] \right\}$. By observation we see that

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{b}\begin{bmatrix} b \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{a}\begin{bmatrix} a \\ a \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}.$$

Putting this all together, we get that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3a}\begin{bmatrix} a \\ a \\ 0 \end{bmatrix} + \frac{1}{3b}\begin{bmatrix} b \\ 0 \\ b \end{bmatrix} + \frac{-1}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) From the row reductions above, we see that

$$\operatorname{row}(A) = \operatorname{span}\left\{ \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ c \\ 0 \\ e \end{bmatrix} \right\}, \qquad \operatorname{null}(A) = \left\{ \begin{bmatrix} -c/a \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -d/b \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -e/a \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

from which we observe that

$$\begin{bmatrix} a \\ 0 \\ c \\ 0 \\ e \end{bmatrix} - c\begin{bmatrix} -c/a \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - e\begin{bmatrix} -e/a \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a + c^2/a + e^2/a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We would like $a + c^2/a + e^2/a \neq 0$, but this is already true, as

$$a + c^2/a + e^2/a = 0 \quad \Longleftrightarrow \quad a^2 + c^2 + e^2 = 0,$$

which can only happen if all of a, c, e are zero, which can not happen. Hence

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{a + c^2/a + e^2/a} \begin{bmatrix} a \\ 0 \\ c \\ 0 \\ e \end{bmatrix} + \frac{-c}{a + c^2/a + e^2/a} \begin{bmatrix} -c/a \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{-e}{a + c^2/a + e^2/a} \begin{bmatrix} -e/a \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

5. Let $S \subseteq \mathcal{M}_{2\times 2}$ be the space of symmetric 2×2 matrices.

(a) Show that $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for S.

(b) Extend B to an orthogonal basis for $\mathcal{M}_{2\times 2}$.

You may use the fact that $\mathcal{M}_{2\times 2} = \text{span}(B')$, where

$$B' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(a) (2 points) This is a basis first because it spans S: any symmetric 2×2 matrix is of the form

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

And it is a basis because the matrix equation

$$x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is equivalent to the four equations

$$\begin{aligned} x + 0 + 0 &= 0 \\ 0 + y + 0 &= 0 \\ 0 + y + 0 &= 0 \\ 0 + 0 + z &= 0, \end{aligned}$$

and the only solution here is $x = y = z = 0$. Hence the matrices are indepedent. Being indepedent and spanning the space means the set is a basis.

(b) (3 points) The set B is orthogonal already, which we can verify by checking traces. To extend it to a basis of $\mathcal{M}_{2\times 2}$, we need one more matrix (since the dimension is 4, as B has 4 elements). Notice that the two matrices missing from B that are in B' are $X = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], Y = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. Let $Z = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Since the sets $\{Z, X\}$ and $\{Z, Y\}$ are both independent, we only have to perform Gram–Schmidt on one of them to get a fourth orthogonal matrix. We choose $\{Z, X\}$:

$$X' = X - \frac{\text{trace}(Z^T X)}{Z^T Z} Z = X - \frac{1}{2} Z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}.$$

Hence B' extends to the orthogonal basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix} \right\}$$

of $\mathcal{M}_{2\times 2}$.

6. Consider the hyperplane $H = \{(x, y, z, w) \ : \ 2x - 4y + 2z - 1w = 0\} \subseteq \mathbf{R}^4$.

(a) Give a basis for this subspace.

(b) Find two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{R}^4$ not in H, for which:

- the projection of $\mathbf{a}$ onto H is not the same as the projection of $\mathbf{b}$ onto H, and
- the error in projecting both $\mathbf{a}$ and $\mathbf{b}$ onto H is 5.

(a) (2 points) The given equation is the nullspace of the matrix

$$A = \begin{bmatrix} 2 & -4 & 2 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -2 & 1 & -1/2 \end{bmatrix}.$$

Hence the nullspace of A, and the basis for H, is

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) (3 points) Note that the normal vector to H is $(2, -4, 2, -1)$, which has length $\sqrt{4 + 16 + 4 + 1} = \sqrt{25} = 5$. Take $\mathbf{a}$ to be this normal vector, whose projection will have error 5. For $\mathbf{b}$, note that $(1, 0, -1, 0) \in H$, and adding the normal vector to $\mathbf{b}$ we will get a different vector also with projection error 5:

$$\mathbf{a} = \begin{bmatrix} 2 \\ -4 \\ 2 \\ -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 1 \\ -1 \end{bmatrix}.$$

We see that $\text{proj}_H(\mathbf{a}) = 0$ and $\text{proj}_H(\mathbf{b}) = (1, 0, -1, 0)$, which are different points.

7. Consider the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 3 & 2 & 3 \end{bmatrix}$.

(a) Give the LU-decomposition of A.

(b) Give the 2nd row of R in the QR-decomposition of A.

(a) (2 points) To get to upper triangular form, we do two row operations:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We know the product of elementary matrices just combines the off-diagonal entries, and the inverse is the same, but with negative off diagonal entries. Hence the LU-decomposition is

$$\underbrace{\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 3 & 2 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{U}.$$

(b) (3 points) The QR-decomposition uses the columns $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ of A (which are independent, as there are 3 pivots) to create the orthonormal columns $\mathbf{q}_1\mathbf{q}_2, \mathbf{q}_3$ of Q. Since we are only asked for the second row of R, we only need $\mathbf{q}_2$, which is defined using $\mathbf{q}_1$:

$$R = \begin{bmatrix} \mathbf{q}_1^T\mathbf{c}_1 & \mathbf{q}_1^T\mathbf{c}_2 & \mathbf{q}_1^T\mathbf{c}_3 \\ 0 & \mathbf{q}_2^T\mathbf{c}_2 & \mathbf{q}_2^T\mathbf{c}_3 \\ 0 & 0 & \mathbf{q}_3^T\mathbf{c}_3 \end{bmatrix}$$

First we make the orthogonal vectors $\mathbf{w}_1$, $\mathbf{w}_2$.

- $\mathbf{w}_1 = \mathbf{c}_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$
- $\mathbf{w}_2 = \mathbf{c}_2 - \mathrm{proj}_{\mathbf{w}_1}(\mathbf{c}_2) = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} - \frac{\mathbf{c}_2^T\mathbf{w}_1}{\mathbf{w}_1^T\mathbf{w}_1}\mathbf{w}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} - \frac{6}{18}\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-1 \\ -1-0 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Note that $\|\mathbf{w}_2\| = \sqrt{3}$, so $\mathbf{q}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$. Finally we get to row 2 of R:

$$\begin{bmatrix} 0 & \mathbf{q}_2^T\mathbf{c}_2 & \mathbf{q}_2^T\mathbf{c}_3 \end{bmatrix} = \begin{bmatrix} 0 & 3/\sqrt{3} & 2/\sqrt{3} \end{bmatrix}.$$

8. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$.

(a) How many nonzero terms are in the permutation formula of $\det(A)$?

(b) Express (do not evaluate) the determinant of A using the recursive formula.

(c) Use the pivot definition of the determinant to evaluate your answer from part (b).

(a) (2 points) There is only one. Choosing column 3 from row 3 means we must choose column 1 from row 6, and then column 6 from row 1 (to not get 0). This leaves column 2 for row 5 and column 5 for row 2 as the only options for a nonzero term.

(b) (2 points) We choose row 3 because it has many zeros:

$$\det(A) = (-1)^{3+3} \cdot 1 \cdot \det(A^{33}) = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

(c) (1 point) Row reduction gives us:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Hence $\det(A) = 1 \cdot 1 \cdot (-1) \cdot (-1) = 1$.