Final

Introduction to Linear Algebra

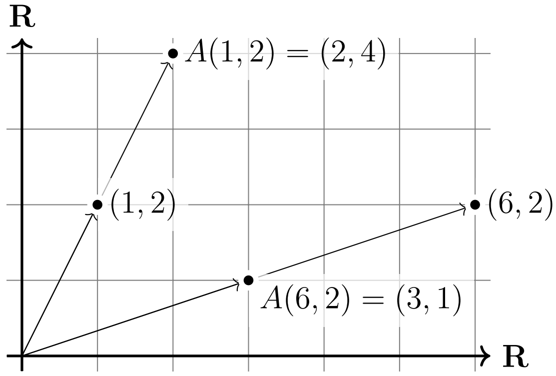
Material from Lectures 13 - 24

Fall 2021

- This final has 5 questions. Each question is worth 5 points. Your answers require justification to recieve points.
- Your grade will be the sum of the 4 highest graded questions. That is, the lowest scoring question will be dropped.
- This is an open-book exam. All work submitted must be your own.
- Write your answer for each question on a separate page. Do not answer more than one question on a single page.
- Submit this final on ORTUS by Monday, December 20, 23:59.

Question	Grade
1	
2	
3	
4	
5	
Total	/20

1. Let $A\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation for which $A\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix}6\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$. This is described in the picture below.



(a) Find the values of $A\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

(b) Construct the matrix of A.

(c) Without computing the inverse of A, explain why A is invertible.

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2. Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathcal{M}_{2\times 2}(\mathbf{Z})$.

   (a) If λ is an eigenvalue of A, find an eigenvalue of $A - \lambda I$.

   (b) If $c = 0$ and $a = d$, explain why A is not diagonalizable.

   (c) If $a = d = 1$ and $c = d$, find a diagonal matrix D similar to A, and find the matrix B for which $D = BAB^{-1}$.

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3. Let $B = \begin{bmatrix} \square & 0 & 0 \\ \square & \square & -1 \\ \square & \square & 1 \end{bmatrix} \in \mathcal{M}_{3\times 3}(\mathbf{Z})$ be a symmetric matrix.

   (a) Fill in the empty entries $\square$ for B, knowing that:
   - $\text{trace}(B) = 5$,
   - $\det(B) = 2$,
   - the two missing entries on the diagonal are different.

   (b) Find the eigenvalues and eigenvectors of B.

   (c) Find one possible $A \in \mathcal{M}_{3\times 3}(\mathbf{R})$ for which $A^T A = B$.

   (d) Using the eigenvectors of B as the right singular vectors of A, give the singular value decomposition of A.

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4. Let $T\colon \mathbf{R}^4 \to \mathbf{R}^3$ be a linear transformation of rank 2.

(a) Is T injective? Is T surjective?

(b) If all you know is that $\ker(T) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and that $\operatorname{im}(T) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$, find one possible matrix for T.

(c) Let $S\colon \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation given by

$$S\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad S\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Use your T from part (b) to find the matrix of $L\colon \mathbf{R}^4 \to \mathbf{R}^2$ for which $SL = T$.

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5. (a) For every $n \in \mathbf{N}$, construct a directed, simple, connected graph $G = (V, E)$ with $u, v \in V$, that has exactly one walk of length k from u to v, for every $k = 1, \ldots, n$.

(b) For every $n \in \mathbf{N}_{\geqslant 3}$, construct an undirected, simple, connected graph $G = (V, E)$ that has exactly $n!$ spanning trees.

(c) Let $G = (V, E)$, with $V = \{(a, b) \ : \ a, b \in \{1, 2, 3\}\}$ be the undirected graph defined by

$$\{(x, y), (z, w)\} \in E \quad \Longleftrightarrow \quad (z - x)^2 + (w - y)^2 = 5.$$

Draw G and explain why it is not possible to interpret the transition probability matrix of G (for any chosen edge directions) as a Markov matrix.
