Final

Introduction to Linear Algebra

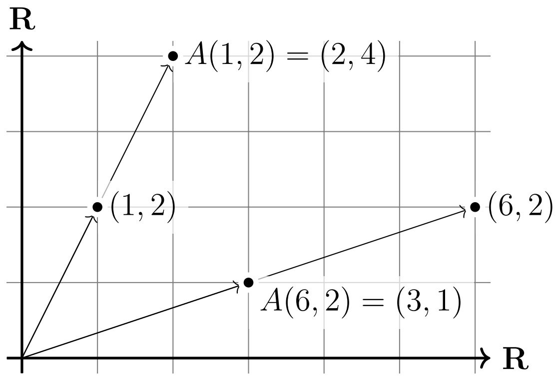
Material from Lectures 13 - 24

Fall 2021

- This final has 5 questions. Each question is worth 5 points. Your answers require justification to recieve points.
- Your grade will be the sum of the 4 highest graded questions. That is, the lowest scoring question will be dropped.
- This is an open-book exam. All work submitted must be your own.
- Write your answer for each question on a separate page. Do not answer more than one question on a single page.
- Submit this final on ORTUS by Monday, December 20, 23:59.

Question	Grade
1	
2	
3	
4	
5	
Total	/20

1. Let $A\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation for which $A\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix}6\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$. This is described in the picture below.



(a) Find the values of $A\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

(b) Construct the matrix of A.

(c) Without computing the inverse of A, explain why A is invertible.

(a) We construct $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ from the given vectors, by clearing the rows with zeros:

$$\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}6\\2\end{bmatrix} = \begin{bmatrix}-5\\0\end{bmatrix} \quad\Longrightarrow\quad \begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{5}\begin{bmatrix}6\\2\end{bmatrix} - \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix},$$

$$\begin{bmatrix}6\\2\end{bmatrix} - 6\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\-10\end{bmatrix} \quad\Longrightarrow\quad \begin{bmatrix}0\\1\end{bmatrix} = \frac{6}{10}\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{10}\begin{bmatrix}6\\2\end{bmatrix}.$$

From this, and by linearity of A, we see what A does to the two standard basis vectors of $\mathbf{R}^2$:

$$A\begin{bmatrix}1\\0\end{bmatrix} = A\left(\frac{1}{5}\begin{bmatrix}6\\2\end{bmatrix} - \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}\right) = \frac{1}{5}A\begin{bmatrix}6\\2\end{bmatrix} - \frac{1}{5}A\begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}3\\1\end{bmatrix} - \frac{1}{5}\begin{bmatrix}2\\4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1\\-3\end{bmatrix},$$

$$A\begin{bmatrix}0\\1\end{bmatrix} = A\left(\frac{6}{10}\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{10}\begin{bmatrix}6\\2\end{bmatrix}\right) = \frac{6}{10}A\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{10}A\begin{bmatrix}6\\2\end{bmatrix} = \frac{6}{10}\begin{bmatrix}2\\4\end{bmatrix} - \frac{1}{10}\begin{bmatrix}3\\1\end{bmatrix} = \frac{1}{10}\begin{bmatrix}9\\23\end{bmatrix}.$$

(b) The observations from part (a) are enough to give us the matrix of A, since any vector $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ can be expressed easily interms of the two standrad basis vectors $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = x\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + y\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. This leads us to the following observations:

$$\begin{aligned}
A\begin{bmatrix}x\\y\end{bmatrix} &= A\left(\begin{bmatrix}x\\0\end{bmatrix} + \begin{bmatrix}0\\y\end{bmatrix}\right)\\
&= A\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= xA\begin{bmatrix}1\\0\end{bmatrix} + yA\begin{bmatrix}0\\1\end{bmatrix}\\
&= x\frac{1}{5}\begin{bmatrix}1\\-3\end{bmatrix} + y\frac{1}{10}\begin{bmatrix}9\\23\end{bmatrix}\\
&= \frac{1}{10}\left(x\begin{bmatrix}2\\-6\end{bmatrix} + y\begin{bmatrix}9\\23\end{bmatrix}\right)\\
&= \frac{1}{10}\begin{bmatrix}2 & 9\\-6 & 23\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.
\end{aligned}$$

Hence $A = \frac{1}{10}\begin{bmatrix} 2 & 9 \\ -6 & 23 \end{bmatrix} = \begin{bmatrix} 1/5 & 9/10 \\ -3/5 & 23/10 \end{bmatrix}$.

(c) Here we can use the fact that the two given vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$ are eigenvectors of A, with eigenvalues 2 and $\frac{1}{2}$ respectively. Then we either apply the fact that no zero eigenvalue means rank must be full, or the fact that the determinant is equal to the product of the eigenvalues. In either case, since we do not have a zero eigenvalue, or since the product of 2 and $\frac{1}{2}$ is not zero, we get that A is invertible.

2. Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathcal{M}_{2\times 2}(\mathbf{Z})$.

   (a) If λ is an eigenvalue of A, find an eigenvalue of $A - \lambda I$.

   (b) If $c = 0$ and $a = d$, explain why A is not diagonalizable.

   (c) If $a = d = 1$ and $c = d$, find a diagonal matrix D similar to A, and find the matrix B for which $D = BAB^{-1}$.

(a) Since A has λ as an eigenvalue, there is some vector $\mathbf{v}$ for which $A\mathbf{v} = \lambda\mathbf{v}$. For this same vector, we see that

$$(A - \lambda I)\mathbf{v} = A\mathbf{v} - \lambda\mathbf{v} = \lambda\mathbf{v} - \lambda\mathbf{v} = 0 = 0\mathbf{v}.$$

Hence 0 is an eigenvalue of $A - \lambda I$, with eigenvector $\mathbf{v}$.

(b) If $c = 0$ and $a = d$, then the matrix A has algebraic multiplicity 2 for the eigenvalue $\lambda = a$, as the eigenvalues of an unpper triangular matrix are on its diagonal. It means that

$$a\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ ay \end{bmatrix},$$

which means that $ax = ax + by$ and $ay = ay$. The first equation implies that $by = 0$, so $y = 0$, since b is fixed. Hence we are only allowed to choose x, meaning that $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ is an eigenvector (and all its multiples). In other words, the geometric multiplicity is 1, which is less than the rank of A, so A can not be diagonalized.

(c) To find D, we find the eigenvectors of A. Note that

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & b \\ 1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 - b = \lambda^2 - 2\lambda + 1 - b,$$

and so

$$\lambda = \frac{2 \pm \sqrt{4 - 4 + 4b}}{2} = 1 \pm \sqrt{b}.$$

With this in mind, we solve the eigenvector equation:

$$\begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} = \begin{bmatrix} 1 & b \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + by \\ x + y \end{bmatrix} \implies \lambda x - \lambda y = by - y \implies y = \frac{\lambda x}{b + \lambda - 1} = \frac{(1 \pm \sqrt{b})x}{b \pm \sqrt{b}}.$$

Hence the eigenvector E matrix and its inverse E^{-1} are

$$E = \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{b}}{b+\sqrt{b}} & \frac{1-\sqrt{b}}{b-\sqrt{b}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{\sqrt{b}} & \frac{-1}{\sqrt{b}} \end{bmatrix}, \qquad E^{-1} = \frac{1}{\frac{-1}{\sqrt{b}} - \frac{1}{\sqrt{b}}}\begin{bmatrix} \frac{-1}{\sqrt{b}} & -1 \\ \frac{-1}{\sqrt{b}} & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{b} \\ 1 & -\sqrt{b} \end{bmatrix}.$$

The diagonalization equation then is

$$A = E\underbrace{\begin{bmatrix} 1 + \sqrt{b} & 0 \\ 0 & 1 - \sqrt{b} \end{bmatrix}}_{D}E^{-1} \implies D = E^{-1}AE,$$

so $B = E^{-1} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{b} \\ 1 & -\sqrt{b} \end{bmatrix}$.

3. Let $B = \begin{bmatrix} \square & 0 & 0 \\ \square & \square & -1 \\ \square & \square & 1 \end{bmatrix} \in \mathcal{M}_{3\times 3}(\mathbf{Z})$ be a symmetric matrix.

(a) Fill in the empty entries $\square$ for B, knowing that:

- $\operatorname{trace}(B) = 5$,
- $\det(B) = 2$,
- the two missing entries on the diagonal are different.

(b) Find the eigenvalues and eigenvectors of B.

(c) Find one possible $A \in \mathcal{M}_{3\times 3}(\mathbf{R})$ for which $A^T A = B$.

(d) Using the eigenvectors of B as the right singular vectors of A, give the singular value decomposition of A.

(a) Let $a = B_{11}$ and $b = B_{22}$. Since $\operatorname{trace}(B) = 5$. we have that $a + b + 1 = 5$. Since $\det(B) = 2$ and B is symmetric, expanding the determinant along the first row, we have that $a(b-1) = 2$. Putting these two equation together, we have

$$\begin{aligned} a+b &= 4 \\ ab - a &= 2 \end{aligned} \quad \Longrightarrow \quad a(4-a) - a = 2$$

$$\Longrightarrow \quad a^2 - 3a + 2 = 0$$

$$\Longrightarrow \quad a = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} \in \{1, 2\}.$$

If $a = 1$, then $b = 3$, and if $a = 2$, then $b = 2$. Since $a \neq b$, it must be that $a = 1, b = 3$. Hence

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(b) Since this is a block matrix, we immediately see that $\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$ is an eigenvector with eigenvalue 1. For the other block $C = \left[\begin{smallmatrix} 3 & -1 \\ -1 & 1 \end{smallmatrix}\right]$, we see that

$$\det(C - \lambda I) = (3-\lambda)(1-\lambda) - 1 = 3 - 3\lambda - \lambda + \lambda^2 - 1 = \lambda^2 - 4\lambda + 2,$$

so $\lambda = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$. The eigenvector equation is

$$\begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - y \\ -x + y \end{bmatrix} \quad \Longrightarrow \quad \begin{aligned} (3-\lambda)x - y &= 0, \\ -x + (1-\lambda)y &= 0. \end{aligned}$$

Note that $x \neq 0$, because if $x = 0$, then $y = 0$, and $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ can't be an eigenvector. Letting $x = 1$, we get $y = 3 - \lambda = 1 \mp \sqrt{2}$. Hence the eigenvalue / eigenvector pairs of C are

$$\lambda_1 = 2 + \sqrt{2},\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 - \sqrt{2} \end{bmatrix}, \qquad \lambda_2 = 2 - \sqrt{2},\ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 + \sqrt{2} \end{bmatrix},$$

and so for B we have $1, 2+\sqrt{2}, 2-\sqrt{2}$ as eigenvalues, with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 - \sqrt{2} \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 + \sqrt{2} \end{bmatrix}.$$

(c) To find A, we consider what happens to $A = \left[\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & i \end{smallmatrix}\right]$ when we multiply it by its transpose. We find that

$$A^T A = \begin{bmatrix} a^2+d^2+g^2 & ab+de+gh & ac+df+gi \\ ab+de+gh & b^2+e^2+h^2 & bc+ef+hi \\ ac+df+gi & bc+ef+hi & c^2+f^2+i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} = B.$$

Observing that the entries $B_{12} = B_{13} = B_{21} = B_{31} = 0$, one potential choice of values in A could have $b = c = d = g = 0$, as this would make the mentioned entries 0 in B. This simplifies the matrices to

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{bmatrix}, \qquad B = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & e^2+h^2 & ef+hi \\ 0 & ef+hi & f^2+i^2 \end{bmatrix}.$$

Since we need $f^2+i^2 = 0$, another potential choice is $f = 0$, $i = 1$. This immediately implies that $h = -1$, and so $e^2 = 2$, for which we can choose $e = \sqrt{2}$. Hence we get

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

though this is by no means the only choice.

(d) For the singular value decomposition, we need vectors of length 1. The relationship between the right singular vectors $\mathbf{v}_i$ and left singular vectors $\mathbf{u}_i$ is given by $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$, where the σ_i are the square roots of $1, 2+\sqrt{2}, 2-\sqrt{2}$. Hence we find the left singular vectors of A to be

$$\mathbf{u}_1 = \frac{1}{\sqrt{1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2+\sqrt{2}}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \left(\frac{1}{\sqrt{2(2-\sqrt{2})}} \begin{bmatrix} 0 \\ 1 \\ 1-\sqrt{2} \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 0 \\ \sqrt{2} \\ -\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

$$\mathbf{u}_3 = \frac{1}{\sqrt{2-\sqrt{2}}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \left(\frac{1}{\sqrt{2(2+\sqrt{2})}} \begin{bmatrix} 0 \\ 1 \\ 1+\sqrt{2} \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 0 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Hence we get the singular value decomposition of A to be

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2+\sqrt{2}} & 0 \\ 0 & 0 & \sqrt{2-\sqrt{2}} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2(2-\sqrt{2})}} & \frac{1-\sqrt{2}}{\sqrt{2(2-\sqrt{2})}} \\ 0 & \frac{1}{\sqrt{2(2+\sqrt{2})}} & \frac{1+\sqrt{2}}{\sqrt{2(2+\sqrt{2})}} \end{bmatrix}}_{V^T}.$$

4. Let $T\colon \mathbf{R}^4 \to \mathbf{R}^3$ be a linear transformation of rank 2.

(a) Is T injective? Is T surjective?

(b) If all you know is that $\ker(T) = \operatorname{span}\left\{ \left[\begin{smallmatrix}1\\0\\0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\0\\1\\0\end{smallmatrix}\right] \right\}$ and that $\operatorname{im}(T) = \operatorname{span}\left\{ \left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\2\\2\end{smallmatrix}\right] \right\}$, find one possible matrix for T.

(c) Let $S\colon \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation given by

$$S\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \qquad S\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}.$$

Use your T from part (b) to find the matrix of $L\colon \mathbf{R}^4 \to \mathbf{R}^2$ for which $SL = T$.

(a) The transformation T is injective iff $\ker(T) = \{0\}$, and since $\ker(T) = \operatorname{null}(A)$, where $A \in \mathcal{M}_{3\times 4}$ is the matrix of T, it must be that $\dim(\operatorname{null}(A)) = 0$ for T to be injective. By the rank-nullity theorem, we have that

$$\begin{array}{rcl} \dim(\operatorname{col}(A^T)) = \operatorname{rank}(A) & = & 2 \\ \dim(\operatorname{col}(A^T)) + \dim(\operatorname{null}(A)) & = & 4 \end{array} \quad \Longrightarrow \quad \dim(\operatorname{null}(A)) = 2.$$

Hence $\operatorname{null}(A) \neq \{0\}$, and so $\ker(T) \neq \{0\}$, meaning that T is not injective. For surjectivity, we have that T is surjective if $\dim(\operatorname{im}(T)) = \dim(\mathbf{R}^3) = 3$. By the dimension theorem,

$$4 = \dim(\mathbf{R}^4) = \dim(\ker(T)) + \dim(\operatorname{im}(T)) = 2 + \dim(\ker(T)),$$

which implies that $\dim(\ker(T)) = 2 \neq 3$, and so T is not surjective.

(b) Given what the kernel and image of T are, we make some choice as to where T sends the standard basis vectors of $\mathbf{R}^4$:

$$T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \qquad T\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \qquad T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \qquad T\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}.$$

This gives the matrix of T immediately as $\begin{bmatrix}0 & 1 & 0 & 2\\ 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 2\end{bmatrix}$.

(c) Here we must have $\ker(L) = \ker(T)$, because (the matrix of) S is full rank, which we observe by $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$ being linearly independent, and by $\left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\2\\2\end{smallmatrix}\right]$ being linearly independent. So as above we choose where the basis vectors of $\mathbf{R}^4$ go, to get that

$$L = \begin{bmatrix}0 & 1 & 0 & 1\\ 0 & 1 & 0 & -1\end{bmatrix}.$$

We may check that $SL\mathbf{e}_i = T\mathbf{e}_i$ on all the standars basis vectors $\mathbf{e}_i$ of $\mathbf{R}^4$, as desired.

5. (a) For every $n \in \mathbf{N}$, construct a directed, simple, connected graph $G = (V, E)$ with $u, v \in V$, that has exactly one walk of length k from u to v, for every $k = 1, \ldots, n$.

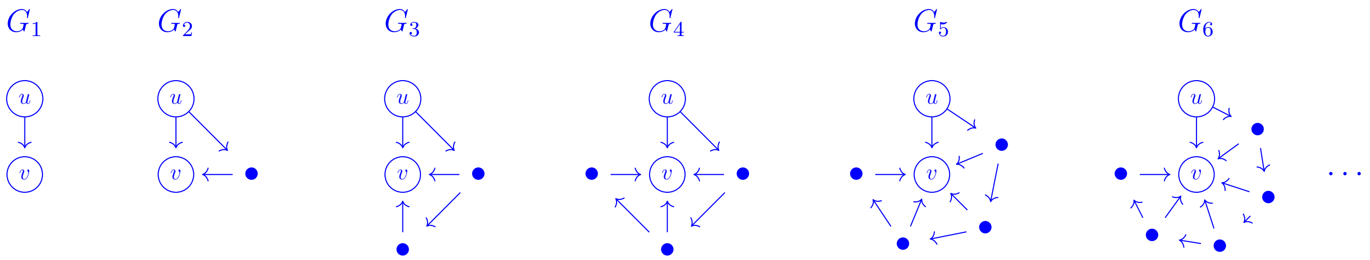
(b) For every $n \in \mathbf{N}_{\geqslant 3}$, construct an undirected, simple, connected graph $G = (V, E)$ that has exactly $n!$ spanning trees.

(c) Let $G = (V, E)$, with $V = \{(a, b) : a, b \in \{1, 2, 3\}\}$ be the undirected graph defined by

$$\{(x, y), (z, w)\} \in E \iff (z - x)^2 + (w - y)^2 = 5.$$

Draw G and explain why it is not possible to interpret the transition probability matrix of G (for any chosen edge directions) as a Markov matrix.

(a) Consider the following sequence of graphs:



Formally, let

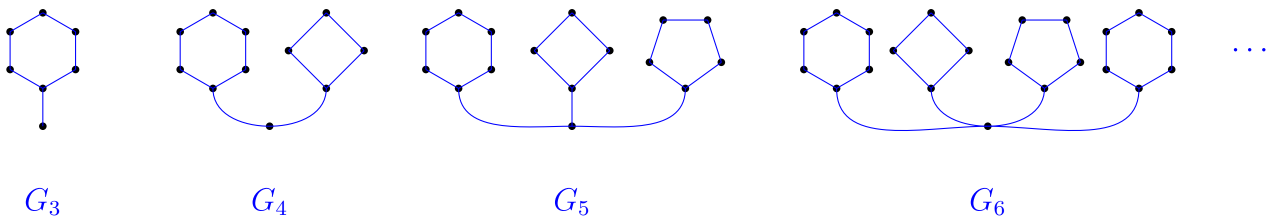
$$\begin{aligned} G_1 &= (V_1 = \{u, v\}, E_1 = \{(u, v)\}), \\ G_2 &= (V_2 = \{u, v, 2\}, E_2 = \{(u, v), (u, 2), (2, v)\}), \\ G_n &= (V_{n-1} \cup \{n\}, E_{n-1} \cup \{(n-1, n), (n, v)\}) \end{aligned}$$

for $n \geqslant 3$. This sequence of graphs satsifies the given condition, where we interpret a walk to be a directed walk.

(b) For this question we use the following two observations:

- The graph C_n which is just a cycle of n edges has n spanning trees
- If $G = (V_G, E_G)$ has k spanning trees and $H = (V_H, E_H)$ has ℓ spanning trees, then the graph with vertex set $V = V_G \cup V_H$ and edge set $E = E_G \cup E_H \cup \{u, v\}$, for $u \in V_G$ and $v \in V_H$, has $k\ell$ spanning trees.

The first observation is quick to verify, and the second follows as $\{u, v\}$ is a cut edge, so must be contained in every tree of the big graph. Hence we consider the following sequence of graphs:



The graph G_3 has $6 = 3!$ spanning trees, G_4 has $4 \cdot 6 = 4!$ spanning trees, G_5 has $5 \cdot 4! = 5!$ spanning trees, and so on.

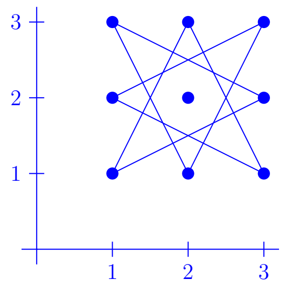
(c) The graph G has 9 vertices. To find all the edges, we note that as a sum of non-negative integers, we can have

$$5 = 0 + 5 \qquad \text{or} \qquad 5 = 1 + 4 \qquad \text{or} \qquad 5 = 2 + 3,$$

but only $5 = 1 + 4$ works as a sum of squares. Hence the edges that exist in G are for $|z - x| = 1$ and $|w - y| = 2$, as well as for $|z - x| = 2$ and $|w - y| = 1$. We quickly find these edges to be as below:

z	x	$\|z - x\|$	w	y	$\|w - y\|$	edge
1	2	1	1	3	2	$(2,3) - (1,1)$
1	2	1	3	1	2	$(2,1) - (1,3)$
2	3	1	1	3	2	$(3,3) - (2,1)$
2	3	1	3	1	2	$(3,1) - (2,3)$
		⋮			⋮	

The other four edges have the positions of z, x swapped with w, y, respectively. Drawing this graph on the Cartesian plane with (x, y) at position (x, y), we get the following graph:



The adjacency matrix of G has 9 rows and 9 columns. If the middle vertex at $(2, 2)$ is in row 5 and column 5, then this row is all zeros and this column is all zeros. Markov matrices must have all rows add up to 1 (right stochastic) or all columns add up to 1 (left stochastic). Neither situation is possible, so this can not be interpreted as a Markov matrix.